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THE MOTION OF A FLUID IN A FIELD OF RADIATION

By E. A. MILNE

[Received 13 January 1930]

1. **Introduction.** The hydrodynamics of a fluid moving in a field of radiation is identical, as regards the dynamical equations, with classical hydrodynamics, provided terms are introduced to allow for the stresses caused by the flux of radiation.* These, in general, form a stress-tensor, partly analogous to the stress-tensor of ordinary viscous motion. The radiative stress-tensor differs from the stress-tensor of viscous motion in that it does not reduce exactly to a simple hydrostatic pressure when the velocity-gradients are put equal to zero; tangential components of stress survive, proportional to the second differential coefficients of the radiant energy-density with respect to the optical thickness.† These terms are, however, negligibly small, under stellar conditions, compared with the principal components of stress, and to the same order the latter are all equal. Approximately, then, if we neglect radiative viscosity and neglect at the same time small terms depending on the second-order temperature gradients, the radiative stresses reduce to a hydrostatic pressure which we shall call p'. All we need to do, *in forming equations of motion*, is to add p' to the ordinary fluid-pressure p. We must not, however, assume without proof that the same rule holds for integrals of the equations, and we shall find later an example (convective equilibrium) where the latter is not true.

The equation of continuity is unaltered.

This scheme of equations suffices for the discussion of a fluid at constant temperature. But when temperature variations exist or are brought about by the motion, and when the heating and cooling effects of the emission and absorption of radiation are taken into account, we require another equation evaluating the rate of change of the temperature at any point. Such an equation is fundamental, for example, in discussing the pulsations of a star or motions in sunspots. The radical importance of this equation can be seen from the

* Rosseland has discussed terms arising from the finite velocity of light, but such terms are small compared with those here considered.

† These are fully discussed by the author in a contribution to *Handbuch der Astrophysik* (in press).

following example. In the classical mechanics of a dynamical system every particle is in equilibrium if at a given instant the accelerations of all the particles are zero. But this condition is not sufficient when temperature changes may occur. If DT/Dt at any point is not zero, where T denotes the temperature and D/Dt is the Lagrangian operator 'following the motion', the resultant heating or cooling may and usually will affect the dynamical stresses, and mechanical equilibrium will not continue to be preserved, in general. If x is a co-ordinate of a particle, D^3x/Dt^3 may be non-zero even if D^2x/Dt^2 and Dx/Dt are zero. It can be shown that the displacement of a given particle will in general obey a differential equation of the third order in the time. This is borne out by Jeans's investigations on the stability of stars.

Unfortunately no consensus of opinion exists as to the form of the equation determining DT/Dt. A fundamental memoir by S. Rosseland,* which derives this equation, has been generally ignored. His method was to evaluate the energy flowing into a closed surface fixed in space and then to apply a Green's theorem transformation. His final equation neglects, however, the mechanical effect of radiation pressure in instantaneously compressing the medium. Eddington† has frequently used an equation claiming to describe the 'adiabatic' motion of a fluid, but his equation treats an element of fluid as if it were enclosed in totally reflecting walls; it ignores the unequal flow of radiant energy through the different parts of the boundary of the element. Jeans‡ has derived an equation for DT/Dt by 'following the motion' of an element as it warms or cools, but he neglected the change in radiant energy imprisoned due to change of volume. His formula was corrected by H. Vogt,§ and the correction was accepted by Jeans,‖ but Vogt's method fails to take into account the difference between the flux of energy through a moving surface and that through a surface at rest.

The following is an attempt to obtain the form of the DT/Dt equation in a rigorous way. The method differs from Rosseland's chiefly in that it evaluates the energy-exchanges of the matter and aether inside a surface moving with the matter instead of inside a fixed surface. The result is an equation different from any yet given; but

* *Astrophys. Journ.* 63, 342 (1926).
† *M.N.* 79, 13 (1918); *Internal Constitution of the Stars*, p. 190 (1926).
‡ *M.N.* 85, 917 (1925).
§ *Ast. Nach.* 232, No. 5545 (1928).
‖ *Astronomy and Cosmogony*, p. 114 (footnote) (1928).

it agrees with Rosseland's when to his equation a radiation-pressure term is added. The accuracy of the equation is checked by using it to calculate the rate of change of the energy of a star which is contracting quasi-statically.

An exact integral of the complete system of equations is then obtained, under certain conditions, in the form of a generalization of the Bernoullian energy-integral of classical hydrodynamics; it gives the change in flux of radiation between any two points on a stream-line when the motion is along the temperature gradient. Lastly an exact integral is obtained for 'convective' or 'adiabatic' equilibrium when account is taken of radiation-pressure. It is usually assumed that under radiation-pressure the well-known adiabatic law $p \propto \rho^{\gamma}$ becomes $p+p' \propto \rho^{\gamma'}$, where ρ is the density, p the gas-pressure, p' the radiation-pressure, γ and γ' ratios of specific heats. This is shown to be false. The temperature distribution in a column in convective equilibrium under gravity and radiation-pressure is obtained, and it is shown that as the depth increases, the density tends to a finite maximum. Some numerical application to solar conditions is made. The temperature distribution in convective equilibrium is shown to be a particular case of the Bernoulli energy-integral.

2. **Lemma.** We shall need the following lemma. Consider a fluid in motion in any manner, either confined within a fixed boundary or with its density vanishing everywhere at the boundary. Let there be no external masses, so that the gravitational potential is derived from the fluid itself. Let V be the gravitational potential at any point, ρ the density, x_α a typical space-co-ordinate (one of x, y, z), u_α the corresponding velocity-component (one of u, v, w), t the time, do an element of volume. *Then*

$$\iiint \rho \frac{\partial V}{\partial t}\, do = \iiint \rho u_\alpha \frac{\partial V}{\partial x_\alpha}\, do,$$

where the volume integrals are taken through the whole fluid and the repetition of the suffix α implies summation over the corresponding three suffixes.*

Proof. By the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\alpha}(\rho u_\alpha) = 0 \qquad (1)$$

* Here and elsewhere.

The potential V at x_α is given by

$$V = \iiint \frac{\rho'}{r}\, do'$$

where $$r^2 = (x-x')^2+(y-y')^2+(z-z')^2$$

and ρ' denotes the density at x'_α. Hence

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \iiint \frac{\partial \rho'}{\partial t}\frac{1}{r}\, do' \\
&= -\iiint \frac{1}{r}\frac{\partial}{\partial x'_\alpha}(\rho' u'_\alpha)\, do' \\
&= -\iiint \frac{\partial}{\partial x'_\alpha}\left(\frac{\rho' u'_\alpha}{r}\right) do' + \iiint \rho' u'_\alpha \frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right) do' \\
&= -\iint \frac{\rho'}{r} n_\alpha u'_\alpha\, dS' + \iiint \rho' u'_\alpha \frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right) do',
\end{aligned}$$

on transforming by Green's theorem and denoting by n_α a typical direction-cosine of the outward normal at the surface element dS'. On the hypotheses made the surface integral vanishes. Hence

$$\iiint \rho \frac{\partial V}{\partial t}\, do = \iiint \rho\, do \iiint \rho' u'_\alpha \frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right) do'.$$

Interchange the order of integration on the right-hand side. Then

$$\begin{aligned}
\iiint \rho \frac{\partial V}{\partial t}\, do &= \iiint \rho' u'_\alpha\, do' \iiint \rho \frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right) do \\
&= \iiint \rho' u'_\alpha\, do' \frac{\partial}{\partial x'_\alpha} \iiint \frac{\rho}{r}\, do \\
&= \iiint \rho' u'_\alpha \frac{\partial V'}{\partial x'_\alpha}\, do' \\
&= \iiint \rho u_\alpha \frac{\partial V}{\partial x_\alpha}\, do. \qquad (2)
\end{aligned}$$

This establishes the lemma.

Corollary. Let dW/dt be the rate of performance of work by the gravitational forces on the fluid. Then since a typical force-component is $\rho \partial V/\partial x_\alpha$,

$$\frac{dW}{dt} = \iiint \rho \frac{\partial V}{\partial x_\alpha} u_\alpha\, do \qquad (3)$$

$$= \tfrac{1}{2}\iiint \rho \frac{\partial V}{\partial x_\alpha} u_\alpha\, do + \tfrac{1}{2}\iiint \rho \frac{\partial V}{\partial t}\, do \qquad (4)$$

on applying the lemma. Multiply the equation of continuity (1) by $\frac{1}{2}V$, integrate over the volume of the fluid, and add to (4). We find

$$\frac{dW}{dt} = \tfrac{1}{2}\iiint\left[\frac{\partial}{\partial t}(\rho V) + \frac{\partial}{\partial x_\alpha}(\rho V u_\alpha)\right] do.$$

But if ϕ is any function of position, by a well-known theorem

$$\frac{D}{Dt}\iiint \phi\, do = \iiint\left[\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x_\alpha}(\phi u_\alpha)\right] do, \tag{5}$$

where D/Dt on the left-hand side denotes differentiating following the moving surface. Hence

$$\frac{dW}{dt} = \frac{D}{Dt}\iiint \tfrac{1}{2}\rho V\, do. \tag{6}$$

By integration of (6) with respect to the time we have a proof of a well-known formula in attractions, namely, that the gravitational potential energy is $-\frac{1}{2}\iiint \rho V\, do$. We must not, however, attach any physical meaning to the *localization* of gravitational energy apparently implied by the existence of this integral; any number of different volume integrals for the energy can be given, each one apparently implying a different localization of energy. What is of physical importance is the localization of the *appearance* of energy when the configuration is changing. This is given by (3) whatever the bounding surface drawn in the fluid. Our result (6) verifies that this localized appearance of energy, integrated over the whole volume of the fluid, is equal to the rate of decrease of the potential energy.

3. **The conservation of energy for a fluid moving under its own stresses, in a field of gravitation and in the presence of radiation.** Let E be the radiant energy per unit volume at any point, $C_v T$ the thermal energy of the fluid per gramme, T the temperature. Then $E+\rho C_v T$ is the total thermal energy contained within unit volume at any point. For perfect gases or for only slightly compressible liquids C_v may be taken as constant or a function of temperature only; it is then the specific heat at constant volume. When dissociation may occur, the above specification may be considered as the definition of C_v. Until we come to discuss perfect gases we shall not need to separate C_v and T.

Let ϵ be the rate of liberation of sub-atomic energy at any point, per gramme. Let p be the hydrostatic pressure, p' the radiation-pressure, F_x, F_y, F_z the components of the net flux of radiation.

The radiation-pressure p' is equal to $\frac{1}{3}E$, but we shall not use this relation until we are compelled, in order to trace the identity of the various terms. Let V be the total gravitational potential, not necessarily wholly due to the fluid under discussion.

Consider a surface S moving with the fluid, i.e. such that it consists, during the motion, of the same fluid-elements, reckoned macroscopically. The rate of change of the energy contained within this surface, thermal and kinetic, is

$$\frac{D}{Dt}\iiint(E+\rho C_v T+\tfrac{1}{2}\rho u_\alpha u_\alpha)\,do\,;$$

The rate of flow of energy into the space enclosed by S consists of four contributions:* (1) the rate of liberation of sub-atomic energy

$$\iiint \epsilon\rho\,do\,;$$

(2) the work done by the total pressure, $p+p'$, at the surface,

$$-\iint(p+p')n_\alpha u_\alpha\,dS\,;$$

(3) the work done by the gravitational forces

$$\iiint \rho u_\alpha \frac{\partial V}{\partial x_\alpha}\,do\,;$$

(4) the rate of flow of radiant energy into the surface. *If the surface were at rest*, the flow would be

$$-\iint n_\alpha F_\alpha\,dS,$$

but the flow through a moving surface will be different. The simplest way of calculating the correction is to notice that as compared with a surface at rest the flow in dt will be greater by the radiant energy contained within the volume through which the surface S expands. Hence there is an added rate of flow of amount

$$\iint En_\alpha u_\alpha\,dS.$$

Hence

$$\frac{D}{Dt}\iiint(E+\rho C_v T+\tfrac{1}{2}u_\alpha u_\alpha)\,do=\iiint\epsilon\rho\,do-\iint(p+p')\,n_\alpha u_\alpha\,dS+$$
$$+\iiint\rho u_\alpha\frac{\partial V}{\partial x_\alpha}\,do-\iint n_\alpha F_\alpha\,dS+\iint En_\alpha u_\alpha\,dS. \quad (7)$$

* We neglect conduction of heat. This could easily be taken into account if necessary, but it is negligible in stellar applications.

It seems at this stage desirable to point out that we must avoid the error of attempting to count in also the rate of change of the *potential energy* of the fluid contained within S. If the surface S contained the whole fluid we could if we liked ignore the work done by the gravitational forces on the volume elements, and calculate instead the rate of increase of the whole potential energy. If the gravitational field were derived entirely from the fluid under consideration, we could delete the term $\iiint \rho u_\alpha(\partial V/\partial x_\alpha)\,do$ on the right-hand side of (7) and add in a term $\dfrac{D}{Dt}\iiint -\frac{1}{2}\rho V\,do$ on the left-hand side. Equation (7), thus altered, should remain valid. The corollary to the lemma shows that this is so. For taking (7) as it stands we can transfer the third integral on the right to the left, obtaining $-\iiint \rho u_\alpha(\partial V/\partial x_\alpha)\,do$, and by (4) and (6) this integral, *when extended to the whole fluid*, is precisely $\dfrac{D}{Dt}\iiint -\frac{1}{2}\rho V\,do$. The upshot of this parenthesis is that in forming (7) we have accurately calculated the rate of liberation of energy, locally, due to changes in the gravitational configuration, whatever the surface S; the quantity ϵ represents sub-atomic liberation only—it does not include gravitational liberation or convective liberation, which are taken into account elsewhere in the equation.

We now make use of the dynamical equations of motion

$$\rho\frac{Du_\alpha}{Dt} = -\frac{\partial(p+p')}{\partial x_\alpha} + \rho\frac{\partial V}{\partial x_\alpha} \tag{8}$$

to remove the terms in $\partial V/\partial x_\alpha$ from (7). We then apply (5) to transform the left-hand side of (7), and we transform the surface integrals on the right-hand side by Green's theorem. The result is

$$\iiint \frac{\partial}{\partial t}[E+\rho C_v T+\tfrac{1}{2}\rho u_\alpha u_\alpha]\,do + \iiint \frac{\partial}{\partial x_\beta}[u_\beta\{E+\rho C_v T+\tfrac{1}{2}\rho u_\alpha u_\alpha\}]\,do =$$

$$= \iiint \epsilon\rho\,do - \iint\!\!\int \frac{\partial}{\partial x_\alpha}[(p+p')u_\alpha]\,do - \iiint \frac{\partial F_\alpha}{\partial x_\alpha}\,do + \iiint \frac{\partial}{\partial x_\alpha}(Eu_\alpha)\,do + \iiint\left[u_\alpha\rho\frac{Du_\alpha}{Dt} + \frac{\partial(p+p')}{\partial x_\alpha}\right]do.$$

Since the surface S is arbitrary, we can now remove the sign of volume integration which occurs throughout. Using

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_\beta \frac{\partial}{\partial x_\beta},$$

and remembering that α and β are dummies, we obtain

$$\frac{\partial E}{\partial t} + C_v T\left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_\beta)}{\partial x_\beta}\right) + \rho \frac{D(C_v T)}{Dt} + \tfrac{1}{2} u_\alpha u_\alpha \left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_\beta)}{\partial x_\beta}\right) =$$

$$= \epsilon\rho - \frac{\partial F_\alpha}{\partial x_\alpha} - (p+p')\frac{\partial u_\alpha}{\partial x_\alpha}.$$

Applying the equation of continuity (1) and using it also in the form

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_\alpha}{\partial x_\alpha} = 0,$$

we have finally

$$\frac{\partial E}{\partial t} + \rho \frac{D(C_v T)}{Dt} - \frac{p+p'}{\rho}\frac{D\rho}{Dt} = \epsilon\rho - \operatorname{div} \mathbf{F}. \tag{9}$$

Equation (9) is the desired equation for DT/Dt which completes the system of equations determining the motion. It should be noticed that E is preceded by the Eulerian operator $\partial/\partial t$, not the Lagrangian D/Dt. This is important in applications. Explicit reference to gravitational energy has disappeared, but it is fully counted in. The equation contains no reference to kinetic energy, though it is by no means obvious, when gravitational contraction is occurring, that no interchange occurs between kinetic energy and thermal energy; the method of proof of Jeans and Vogt, even if it led to the correct result, seems open to objection in this respect.

4. We first give a physical meaning to formula (9). Integrate both sides over any finite volume surrounded by a *fixed* surface. The right-hand side transforms to

$$\iiint \epsilon\rho \, d\sigma - \iint n_\alpha F_\alpha \, dS. \tag{10}$$

This represents the total output of sub-atomic energy less the amount leaving the surface in the form of radiation. It follows that the quantity

$$\frac{\partial E}{\partial t} + \rho \frac{D(C_v T)}{Dt} - \frac{p+p'}{\rho}\frac{D\rho}{Dt}. \tag{11}$$

represents the net gain of liberated sub-atomic energy per unit volume less the net loss by radiation.

5. **Quasi-static contraction of a star.** We apply this to a spherical star, contracting quasi-statically. From the result of the last paragraph, the net rate of gain of energy in the whole star is

$$\iiint\left(\frac{\partial E}{\partial t}+\rho\frac{D(C_v T)}{Dt}-\frac{p+p'}{\rho}\frac{D\rho}{Dt}\right)do.$$

This should be equal to the rate of change of the energy of the star calculated in the usual manner. I have shown elsewhere* that the gravitational energy of the star is

$$-3\iiint(p+p')\,do. \tag{12}$$

The thermal energy is

$$\iiint(E+\rho C_v T)\,do. \tag{13}$$

Now we have always $E=3p'$. Hence, adding (12) and (13) the total energy is

$$\iiint(\rho C_v T-3p)\,do. \tag{14}$$†

Accordingly, for a spherical star contracting quasi-statically we ought to have

$$\frac{D}{Dt}\iiint(\rho C_v T-3p)\,do=\iiint\left(\frac{\partial E}{\partial t}+\rho\frac{D(C_v T)}{Dt}-\frac{p+p'}{\rho}\frac{D\rho}{Dt}\right)do. \tag{15}$$

The satisfaction of this identity will be a test of the correctness of the fundamental equation (9). To verify it, we write

$$\frac{\partial E}{\partial t}=\frac{DE}{Dt}-u\frac{\partial E}{\partial r}, \tag{16}$$

where u is the (slow) radial velocity at distance r from the centre. Since $do=4\pi r^2\,dr$, we have on integrating by parts

$$-\iiint u\frac{\partial E}{\partial r}\,do=[-4\pi r^2 uE]+\int E\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)4\pi r^2\,dr.$$

The integrated part vanishes at both limits, neglecting, as in the

* *M.N.* 89, 740 (1929), equation (6).

† We notice that for a star of perfect gas, obeying the equation of state $p=(R/\mu)\rho T$, for which $C_v(\gamma-1)=R/\mu$, γ being the ratio of specific heats, expression (14) reduces to

$$\iiint\frac{4-3\gamma}{\gamma-1}p\,do.$$

This expression, though containing no explicit reference to the radiant energy, fully includes it. It remains true if γ and μ are variable from place to place in the star.

demonstration of (12), the relatively small surface-value of E. The equation of continuity is here

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\frac{2u}{r}+\frac{\partial u}{\partial r}=0. \tag{17}$$

Hence the right-hand side of (15) becomes

$$\int\left(\frac{DE}{Dt}-\frac{E}{\rho}\frac{D\rho}{Dt}+\rho\frac{D(C_v T)}{Dt}-\frac{p+p'}{\rho}\frac{D\rho}{Dt}\right)4\pi r^2\,dr. \tag{18}$$

Now in quasi-static motion we have

$$\frac{1}{\rho}\frac{\partial(p+p')}{\partial r}=-\frac{GM(r)}{r^2}, \tag{19}$$

where G is the constant of gravitation and $M(r)$ is the mass contained in radius r. Apply the operator D/Dt to this equation. We find

$$-\frac{1}{\rho^2}\frac{\partial(p+p')}{\partial r}\frac{D\rho}{Dt}+\frac{1}{\rho}\frac{\partial}{\partial r}\frac{D(p+p')}{Dt}-\frac{1}{\rho}\frac{\partial(p+p')}{\partial r}\frac{\partial u}{\partial r}=\frac{GM(r)}{r^3}2u. \tag{20}$$

Substitute for $GM(r)/r^2$ from (19) in (20) and use (17). We find

$$\frac{\partial}{\partial r}\frac{D(p+p')}{Dt}+\frac{4u}{r}\frac{\partial(p+p')}{\partial r}=0. \tag{21}$$

Multiply (21) by $4\pi r^3\,dr$, and integrate through the star. Integrating by parts and again using the equation of continuity (17) we find

$$3\int\frac{D(p+p')}{Dt}4\pi r^2\,dr-4\int\frac{p+p'}{\rho}\frac{D\rho}{Dt}4\pi r^2\,dr=0. \tag{22}$$

In (18) replace the term in DE/Dt by the term in $3Dp'/Dt$ given by (22). Then (18) becomes

$$\int\left(3\frac{p+p'}{\rho}\frac{D\rho}{Dt}-\frac{E}{\rho}\frac{D\rho}{Dt}-3\frac{Dp}{Dt}+\rho\frac{D(C_v T)}{Dt}\right)4\pi r^2\,dr$$

or

$$3\int\left(\frac{p}{\rho}\frac{D\rho}{Dt}-\frac{Dp}{Dt}\right)do+\int\rho\frac{D(C_v T)}{Dt}\,do. \tag{23}$$

In the first of these integrals substitute for $D\rho/Dt$ from the equation of continuity. It becomes

$$-3\int\left(\frac{Dp}{Dt}+p\,\mathrm{div}\,\mathbf{u}\right)do,$$

$\mathbf{u}$ being the vector velocity; this is simply

$$-\frac{3D}{Dt}\int p\,do. \tag{24}$$

Since ρdo is conserved during the motion, the second integral in (23) may be written

$$\frac{D}{Dt}\int \rho C_v T\, do. \tag{24'}$$

Adding (24) and (24') we find that the right-hand side of (15) is

$$\frac{D}{Dt}\int (\rho C_v T - 3p)\, do,$$

which is precisely the left-hand of (15). This check is one which any proposed equation of the type (9) must satisfy.

6. **Comparison with results of other investigators.** The formula of Jeans,* as corrected by Vogt,† runs in the present notation

$$\frac{DE}{Dt} + \rho \frac{D(C_v T)}{Dt} - \frac{(p+p'+E)}{\rho}\frac{D\rho}{Dt} = \epsilon\rho - \operatorname{div}\mathbf{F}. \tag{9'}$$

This differs from our result (9). But it will be observed that the left-hand side of (9') is identical with the integrand of (18). It follows that (9') satisfies the test we have just been applying. Thus (9') is correct on the average, for a star contracting quasi-statically. Equation (9') corresponds to a different partition of the radiant energy amongst the volume elements. Reference to Vogt's proof will show that in the deduction of (9') the flow of radiation through a moving surface has been taken as identical with that through a fixed surface.

Rosseland's‡ formula in the present notation is

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial t}(\rho C_v T) + \operatorname{div}(\rho C_v T\mathbf{u}) + p \operatorname{div}\mathbf{u} = \epsilon\rho - \operatorname{div}\mathbf{F}. \tag{9''}$$

Introduction of the operator D/Dt and use of the equation of continuity converts (9'') into (9) provided p is replaced by $p+p'$. In Rosseland's derivation, however, p denotes (as with us) the 'hydrostatic part' of the material stress-tensor, so that the mechanical effect of the radiation-pressure has been ignored.

I have found the simple form (9) more convenient than Rosseland's in making actual calculations.

7. **The analogue of the Bernoulli energy-integral.** Consider now the case of *steady motion* in a region in which the liberation of

* *M.N.* 85, 917 (1925), equation (6); see also *Astronomy and Cosmogony*, p. 114 (1928), equation (105–4) and footnote.

† *Ast. Nach.* 232, No. 5545 (1928), equation (11).

‡ *Astrophys. Journ.* 63, 355 (1926), equation (37).

sub-atomic energy ϵ is zero. Then since E at any point is now constant in time, $\partial E/\partial t$ is zero, and (9) reduces to

$$\rho\frac{D(C_v T)}{Dt}-\frac{p+p'}{\rho}\frac{D\rho}{Dt}=-\operatorname{div}\mathbf{F}. \tag{25}$$

Now consider the special case in which the streamlines coincide with the curves of flux defined by the vector $\mathbf{F}$, i.e. the special case in which the flow is along the temperature-gradient. (The upward flow in a sun-spot is of this type.) Let σ be the cross-sectional area of a given elementary tube of flow at any point, s the arc measured along the axis of the tube. Then if q denotes the scalar velocity of the fluid, $Dt = ds/q$, and

$$\operatorname{div}\mathbf{F}=\frac{1}{\sigma}\frac{d}{ds}(F\sigma),$$

where $F = |\mathbf{F}|$. Hence (25) becomes

$$q\left[\rho\frac{d(C_v T)}{ds}-\frac{p+p'}{\rho}\frac{d\rho}{ds}\right]=-\frac{1}{\sigma}\frac{d}{ds}(F\sigma). \tag{26}$$

Now the equation of continuity is here

$$\rho q\sigma = \text{constant} = f, \text{ say}.$$

Hence (26) may be written

$$f\left[\frac{d(C_v T)}{ds}-\frac{p+p'}{\rho^2}\frac{d\rho}{ds}\right]=-\frac{d}{ds}(F\sigma). \tag{27}$$

Now the equation of motion is here

$$\rho q\frac{dq}{ds}=\rho\frac{dV}{ds}-\frac{d(p+p')}{ds}$$

or

$$q\frac{dq}{ds}=\frac{dV}{ds}-\frac{d}{ds}\left(\frac{p+p'}{\rho}\right)-\frac{p+p'}{\rho^2}\frac{d\rho}{ds}. \tag{28}$$

Multiplying (28) by f and adding to (27) we find

$$f\left[\frac{d(C_v T)}{ds}+q\frac{dq}{ds}-\frac{dV}{ds}+\frac{d}{ds}\left(\frac{p+p'}{\rho}\right)\right]=-\frac{d}{ds}(F\sigma). \tag{29}$$

This integrates in the form

$$f\left[C_v T+\tfrac{1}{2}q^2-V+\frac{p+p'}{\rho}\right]+F\sigma=C, \tag{30}$$

where C is constant along the streamline. Dividing by f and substituting its value, (30) becomes

$$C_v T+\tfrac{1}{2}q^2-V+\frac{p+p'}{\rho}+\frac{F}{\rho q}=C', \tag{31}$$

where C' is constant along the streamline. Equation (31) is the analogue of the Bernoullian energy-integral of classical hydrodynamics when account is taken of the flux of radiation and radiation-pressure.

Equation (31) may be considered a fundamental relation in the theory of sun-spots. Here we may consider the motion rectilinear and vertical, approximately, and so here ρq is constant. Also $dV/dx = -g$, x being the height, so that $V = -gx$. Thus (31) becomes

$$q[\rho C_v T + \tfrac{1}{2}\rho q^2 + g\rho x + p + p'] + F = \text{constant}. \tag{32}$$

This equation expresses the constancy of the sum of the fluxes of energy of all kinds past any point.* Thus it determines the difference in radiation-flux between the top and bottom of a sun-spot. An approximate form of (32) has been given by Rosseland,† and applied by him to sun-spots, but Rosseland only stated the result in a form involving undefined means. The numerical application of (32) to sun-spots will be given elsewhere.

8. **Adiabatic motion in a star.** For a spherically symmetrical star we may substitute for $\partial E/\partial t$ in terms of DE/Dt by means of (16). Equation (9) then becomes

$$\frac{DE}{Dt} - u\frac{\partial E}{\partial r} + \rho\frac{D(C_v T)}{Dt} - \frac{p+p'}{\rho}\frac{D\rho}{Dt} = \epsilon\rho - \operatorname{div}\mathbf{F}. \tag{33}$$

If we neglect the heating and cooling by the absorption and emission of radiation, and the heating by liberation of sub-atomic energy, in comparison with the changes of temperature produced by convection and by compression and expansion, we have what Eddington calls the 'adiabatic' motion of stellar material. The assumption is equivalent to putting the right-hand side of the fundamental equation (9) equal to zero, and accordingly (33) becomes

$$\frac{DE}{Dt} - u\frac{\partial E}{\partial r} + \rho\frac{D(C_v T)}{Dt} - \frac{p+p'}{\rho}\frac{D\rho}{Dt} = 0. \tag{34}$$

Describe by δ the changes occurring in an interval δt following the motion. Then $u\delta t = \delta r$, the radial expansion, and (34) becomes

$$\delta E - \frac{\partial E}{\partial r}\delta r + \rho\delta(C_v T) - \frac{p+p'}{\rho}\delta\rho = 0. \tag{35}$$

This equation may be compared with the one given by Eddington for adiabatic motion. Considering each element of material as if

* As does (30).

† Loc. cit., equation (53).

enclosed 'adiabatically', i.e. in totally reflecting walls, he finds* in our notation

$$\delta\left[\frac{C_v\rho T+E}{\rho}\right]+(p+p')\delta\left(\frac{1}{\rho}\right)=0. \tag{36}$$

This gives, on multiplication by ρ,

$$\delta E+\rho\delta(C_v T)-\frac{p+p'+E}{\rho}\delta\rho=0. \tag{35'}$$

This differs from (35). Equation (35') is equivalent to the Jeans-Vogt equation (9') for the case of adiabatic motion, and so gives the right change in energy when integrated over the whole star. But it corresponds to a faulty allocation of radiant energy amongst the volume elements. It assumes in effect convection of radiant energy with the material. But we cannot legitimately regard the volume elements as enclosed in totally reflecting walls. Equation (35) shows that Eddington's formula requires correction for the gradient of energy-density.

The simplest way of seeing that the Jeans-Vogt formula, and with it Eddington's formula, implies convection of radiant energy with the material is to derive the Bernoullian energy-integral on the basis of the Jeans-Vogt formula. If we repeat the analysis of section 7, using (9') instead of (9), we find for the integral

$$f\left[\frac{E}{\rho}+C_v T+\tfrac{1}{2}q^2-V+\frac{p+p'}{\rho}\right]+F\sigma=C \tag{30'}$$

instead of (30). The term fE/ρ would represent convection of radiant energy σqE. Actually under the conditions of the problem the distribution of radiant energy remains stationary; there is no *convection* of radiant energy, but simply the radiation flux $F\sigma$. Equation (30) could in fact be written down from first principles; the importance of the formal proof of section 7 is that it confirms the correctness of (9) and disproves the Jeans-Vogt formula. The significant point is the occurrence of $\partial/\partial t$, not D/Dt, in front of E in (9), and it is strong support for the correctness of (9) that Rosseland's method, using a *fixed* surface, also yields $\partial E/\partial t$.

9. **Convective equilibrium under radiation-pressure.** Our last application of (9) is to the type of distribution which is known (curiously enough) as 'convective equilibrium'. This is the type of

* *Internal Constitution of the Stars*, p. 190 (1926), equation preceding (129—2). We have simply replaced his V, which may be taken to be the specific volume, by $1/\rho$.

equilibrium when the material is being constantly stirred in such a way that the resulting convection currents are so rapid that the heating and cooling by compression and expansion is dominant compared with the heating and cooling by absorption and emission of radiation.*

Since the state is steady, $\partial E/\partial t = 0$, and (9) reduces to

$$\rho \frac{D(C_v T)}{Dt} - \frac{p+p'}{\rho}\frac{D\rho}{Dt} = \epsilon\rho - \operatorname{div} \mathbf{F}. \tag{37}$$

If q is the velocity, Ds the element of path of any element during the stirring, $Dt = Ds/q$ and (37) becomes

$$\rho \frac{D(C_v T)}{Ds} - \frac{p+p'}{\rho}\frac{D\rho}{Ds} = \frac{1}{q}(\epsilon\rho - \operatorname{div} \mathbf{F}). \tag{38}$$

On the hypothesis of convective equilibrium, q for any small element is relatively so large that the right-hand side of this equation may be neglected in comparison with the left-hand side. Also Ds is random in direction. Hence (38) reduces to

$$\rho d(C_v T) - \frac{p+p'}{\rho} d\rho = 0. \tag{39}$$

Equation (39) is the fundamental differential equation defining convective equilibrium. It needs of course to be combined with the equation of mechanical equilibrium of the material as a whole before the space-temperature distribution becomes definite. But equation (39) as it stands defines a relation between ρ and T for any given equation of state. It is usually assumed without proof, on the analogy of the corresponding differential equation when $p' = 0$, that in the case of a perfect gas the integral of (39) is of the form

$$p+p' \propto \rho^{\gamma'}, \tag{40}$$

where γ' is a suitably defined ratio of specific heats. I propose to show that this assumption is false. We proceed to integrate the differential equation (39) when the equation of state is that of a perfect gas.

We have always $$p' = \tfrac{1}{3}aT^4. \tag{41}$$

* 'Stirring' of course implies the introduction of external forces not taken into account in deducing (9). But other definitions of convective equilibrium which may be framed (e.g. the state *after* stirring) will be found to imply (39). The method of deduction of (39) from (9) is due to Rosseland, loc. cit., p. 355.

For a perfect gas, with ratio of specific heats γ, gas-constant R, and molecular weight μ, we have the equations

$$p = (R/\mu)\rho T, \tag{41'}$$

$$C_v = \frac{R}{(\gamma-1)\mu}. \tag{41''}$$

Inserting these in (39) we find

$$\frac{R}{(\gamma-1)\mu}\rho dT - \left[\frac{R}{\mu}T + \frac{\frac{1}{3}aT^4}{\rho}\right]d\rho = 0$$

or

$$\frac{R}{\mu}\left[\frac{1}{\gamma-1}\frac{\rho dT}{T^4} + \frac{d\rho}{T^3}\right] = \tfrac{1}{3}a\frac{d\rho}{\rho}. \tag{42}$$

Multiply both sides by $\rho^{3\gamma-4}$. Then both sides become perfect differentials and we find

$$-\frac{R}{3(\gamma-1)\mu}d\left(\frac{\rho^{3\gamma-3}}{T^3}\right) = \frac{\frac{1}{3}a}{3\gamma-4}d(\rho^{3\gamma-4}).$$

Integrating we have

$$\frac{R}{3(\gamma-1)\mu}\left(\frac{\rho^{\gamma-1}}{T}\right)^3 + \frac{\frac{1}{3}a}{3\gamma-4}\rho^{3\gamma-4} = C, \tag{43}$$

where C is a constant.

By means of (41) and (41'), (43) may be put in either of the equivalent forms

$$\frac{1}{3(\gamma-1)}\frac{p}{p'} + \frac{1}{3\gamma-4} = \frac{C}{\frac{1}{3}a}\frac{1}{\rho^{3\gamma-4}} \tag{44}$$

$$\frac{1}{3(\gamma-1)} + \frac{1}{3\gamma-4}\frac{p'}{p} = C\left(\frac{\mu}{R}\right)^4\left(\frac{p}{\rho^\gamma}\right)^3. \tag{44'}$$

No value of γ' exists, either a constant or a function of T, for which (44) or (44') can be put in the form (40).

The case of negligible radiation-pressure is obtained by putting $a = 0$ in (43) or $p' = 0$ in (44'). We have then respectively

$$T \propto \rho^{\gamma-1},$$

$$p \propto \rho^\gamma,$$

the usual results. We see that the presence of radiation-pressure makes a fundamental difference to the integrated adiabatic relation; we cannot simply add the radiation-pressure to the gas-pressure.

If $\gamma > \frac{4}{3}$, (44) shows that p/p' decreases as ρ increases; (44') then shows that p/ρ^γ increases as ρ increases. Thus as we pass down a column in convective equilibrium, radiation-pressure p' becomes

increasingly important relative to gas-pressure p; p/p' is not even approximately constant.

10. **The convective equilibrium of a vertical column of gas under radiation-pressure.** An atmospheric column of perfect gas in convective equilibrium under gravity, in the absence of radiation-pressure, is well known to possess a temperature distribution in which the temperature decreases linearly with the height. We proceed to determine the change brought about by radiation-pressure.

The equation of mechanical equilibrium for a vertical column under gravity g is

$$\frac{d(p+p')}{dx} = -g\rho, \tag{45}$$

where x is the height measured vertically upwards. This may be written

$$\frac{d}{dx}\left(\frac{p+p'}{\rho}\right)+\frac{p+p'}{\rho^2}\frac{d\rho}{dx} = -g. \tag{46}$$

The differential equation of convective equilibrium, (39), may be written in the form

$$\frac{d(C_v T)}{dx}-\frac{p+p'}{\rho^2}\frac{d\rho}{dx} = 0. \tag{47}$$

Adding (46) and (47) we find on integration

$$C_v T+\frac{p+p'}{\rho} = -g(x-A), \tag{48}$$

where A is a constant.

Equation (48) is the general equation giving the space-temperature distribution in convective equilibrium. By its mode of derivation, (48) is seen to be a particular case of the Bernoullian energy-integral (32). For in convective equilibrium the flux of macroscopic kinetic energy is zero, and the radiative flux is disregarded. Removing from (32) the terms in q^2 and F we obtain (48). I do not remember to have seen this aspect of convective equilibrium mentioned previously. It will be noticed that in deriving (48) we have used no equation of state; we have simply used the general differential equation of convective equilibrium, not its perfect-gas integral.

For a perfect gas, the use of (41), (41′), and (41″) transforms (48) into

$$\frac{\gamma}{\gamma-1}\frac{RT}{\mu}+\frac{\frac{1}{3}aT^4}{\rho} = -g(x-A). \tag{49}$$

If in this we neglect the term $\frac{1}{3}aT^4$, we obtain T as a linear function of x, the classical result.

To discuss the form of distribution implied by (49) we now use the integral convective relation (43) in the form

$$\frac{R}{3(\gamma-1)\mu}\frac{\rho}{T^3}+\frac{\frac{1}{3}a}{3\gamma-4}=C\rho^{4-3\gamma}, \tag{50}$$

and substitute for ρ/T^3 in the second term of (49). We find

$$\frac{RT}{\mu(\gamma-1)}\left[\gamma+\frac{\frac{1}{3}}{\frac{C}{\frac{1}{3}a}\rho^{4-3\gamma}-\frac{1}{3\gamma-4}}\right]=-g(x-A). \tag{51}$$

This can also be put in the form

$$\frac{RT}{\mu(\gamma-1)}\left[\frac{4}{3}+\frac{3(\gamma-\frac{4}{3})^2}{(3\gamma-4)-(\frac{1}{3}a/C)\rho^{3\gamma-4}}\right]=-g(x-A). \tag{51'}$$

We may confine our attention to the case $\gamma>\frac{4}{3}$. At great heights, ρ is small, and the temperature distribution is approximately linear in x. As we descend, ρ increases, but it never exceeds the limiting value $\rho_{\max}$ given by

$$\rho_{\max}^{3\gamma-4}=\frac{C}{\frac{1}{3}a}(3\gamma-4). \tag{52}$$

This value is approached asymptotically as $x\to-\infty$. For this limiting value, $\rho^{\gamma-1}/T$ is zero by (43), and so, since ρ is finite, T is infinite. Equation (51′) shows that at great depths T increases with $-x$ less rapidly than linearly.

In terms of $\rho_{\max}$, (44) and (51′) become

$$\frac{3\gamma-4}{3(\gamma-1)}\frac{p}{p'}+1=\left(\frac{\rho_{\max}}{\rho}\right)^{3\gamma-4} \tag{53}$$

$$\frac{RT}{\mu(\gamma-1)}\left[\frac{4}{3}+\frac{\gamma-\frac{4}{3}}{1-(\rho/\rho_{\max})^{3\gamma-4}}\right]=-g(x-A). \tag{54}$$

11. **Numerical application to solar conditions.** A sun-spot is sometimes supposed to be a region of convective equilibrium. It is, however, rather a region of convective *ascent*—mass motion in which cooling by convection occurs according to equation (32). Thus the concept of convective equilibrium does not apply. Convective equilibrium can only be established by some external mixing agency, and it is not clear that any such agency exists on the sun, even in sun-spots. Nevertheless, any turbulent motion will tend to set up a temperature-distribution striving to approach that of convective

equilibrium. Just as in the case of the earth's atmosphere, the actual state of affairs will be intermediate between radiative equilibrium and convective equilibrium. Radiative equilibrium on the sun has been thoroughly studied. It becomes of interest to study numerically the other extreme, convective equilibrium.

For definiteness take $\gamma = \frac{5}{3}$. Then $3\gamma - 4 = 1$ and $3(\gamma - 1) = 2$. Consider a column in convective equilibrium for which the conditions at the top correspond to those in the umbra of a sun-spot. Let the suffix 0 refer to this level. We shall adopt

$$p_0 = 5 \times 10^{-5} \text{ atmos.} = 50 \text{ dynes cm.}^{-2}, \quad T_0 = 4{,}500°.$$

Then if we assume for definiteness a mean molecular weight $\mu = 20$, with $R = 8{\cdot}26 \times 10^7$, we have $\rho_0 = 2{\cdot}69 \times 10^{-9}$ gm. cm.$^{-3}$. Further, with $a = 7{\cdot}64 \times 10^{-15}$, we find $p_0' = 1{\cdot}04$ dyne cm.$^{-2}$. Introducing these values in (53) we find

$$\frac{\rho_{\max}}{\rho_0} = 25, \qquad \rho_{\max} = 6{\cdot}72 \times 10^{-8} \text{ gm. cm.}^{-3}.$$

To obtain an idea of the rate at which ρ approaches $\rho_{\max}$ as we descend, we will now determine the level where $\rho = \frac{1}{2}\rho_{\max}$. Let the suffix $\frac{1}{2}$ denote this level. Then by (53)

$$p_{\frac{1}{2}}/p_{\frac{1}{2}}' = 2,$$

whence, using the value $\rho_{\frac{1}{2}} = \frac{1}{2} \times 6{\cdot}72 \times 10^{-8}$ gm. cm.$^{-3}$, we find $T_{\frac{1}{2}} = 30{,}000°$. To find the depth in kilometres we use (54). The depth $x_0 - x_{\frac{1}{2}}$ is given by

$$\begin{aligned} x_0 - x_{\frac{1}{2}} &= \frac{R}{\frac{2}{3}\mu g}\left[T_{\frac{1}{2}}\left(\frac{4}{3}+\frac{\frac{1}{3}}{\frac{1}{2}}\right) - T_0\left(\frac{4}{3}+\frac{\frac{1}{3}}{24/25}\right)\right] \\ &= 120 \text{ Km.}, \end{aligned}$$

on inserting $g = 2{\cdot}74 \times 10^4$ cm. sec.$^{-2}$.

Thus under the conditions of convective equilibrium, in a depth of 120 Km., with $\mu = 20$, the temperature increases from 4,500° to 30,000°, and the ratio of gas-pressure to radiation-pressure diminishes from 48 to 2. In radiative equilibrium, on the other hand, the ratio of gas-pressure to radiation-pressure changes only very slowly in the photospheric layers of a star.* In the photospheric layers on the sun, for $\mu = 20$ as above, the gradient of temperature is about 150° per Km., and a temperature increase of 25,500° would be only achieved in 170 Km.; the mean gradient in the corresponding convective

* See Bakerian Lecture, *Phil. Trans.* 228A, 456 (1929).

region is about 215° per Km. If the photospheric temperature is taken as 6,000°, the temperature in the convective column will overtake that in the radiative column at a depth of 25 Km., where the temperature is 9,500°. This is where we should place the base of the spot-vortex* if it could be assumed to be in convective equilibrium. Actually, since the gradient in the ascending gases in the spot will be intermediate between radiative and convective equilibrium, the base of the spot-vortex will be deeper than this. Rosseland has given an estimate of 50 Km. for the thickness of the spot-vortex, and this is in agreement with some unpublished calculations of my own. The above calculations are not intended to represent sun-spot conditions—as remarked above the latter will be governed by equation (32)—but they give a concrete picture of convective equilibrium, showing in particular the rapid approach to a limiting density.

12. **Disproof of convective equilibrium in a star as a whole.** If the material forming a star were rapidly stirred throughout, so that convective equilibrium obtained momentarily, ρ and T would be connected by equation (44). Hence nowhere in the star could the density exceed the value $\rho_{\max}$ given by (52). From our numerical work in section 11, a star with a pressure of 5×10^{-5} atmos. in its photosphere and an effective temperature of 4,500° would have a mean density less than 10^{-7} gm. cm.$^{-3}$ for $\mu = 20$ and a still smaller one for smaller μ. The actual numerical values are of small importance, but the mere order of magnitude of the result shows that stars as a whole cannot be in convective equilibrium or in a state in any way approaching it; the observed mean densities are too large. In convective equilibrium they would be orbs of immensely greater radius. Formula (44) affords a justification, in the simplest possible way, of the assumption of radiative equilibrium for the stars as a whole.

13. **Summary.** The fundamental equation governing temperature changes in a compressible medium in motion, taking account of radiation, is obtained in the form of equation (9), section 3. It differs from the form given by previous investigators. The equation is applied to quasi-static motion, to steady motion along temperature gradients, to adiabatic motion, and to convective equilibrium.

I should like to express my general indebtedness to Professor Rosseland's memoir, already cited.

* Cf. H. N. Russell, *Astrophys. Journ.*, 54, 293 (1921).

SOME PROBLEMS CONNECTED WITH FOURIER'S WORK ON TRANSCENDENTAL EQUATIONS

By G. PÓLYA

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THIS paper is concerned with a very bold argument by which Fourier tried to prove that a certain integral function has only real zeros: Fourier applied a rule which was, then and now, proved only for polynomials, to a transcendental function. The question arises whether this application can be justified by some general theorem. This question was much discussed between Fourier and Poisson, mentioned by some contemporary writers, and quite forgotten afterwards. I resume the question in the present paper. I have not been able to solve it completely, though I have been led by it to some general theorems which may be interesting in themselves (section 3). All I can show is that we may draw from Fourier's argument *some* definite conclusion, namely, that the function under consideration has *either* only real zeros *or* an *infinity* of imaginary zeros.

1. **Historical introduction.** The integral function which Fourier tried to prove to have real zeros only is

$$f(x) = 1 - \frac{x}{1^2} + \frac{x^2}{1^2 . 2^2} - \frac{x^3}{1^2 . 2^2 . 3^2} + \ldots = J_0(2\sqrt{x}) \tag{1}$$

(in the notation current now). The algebraical rule to which he appealed is this: *a polynomial $f(x)$ with real coefficients has real roots only, if its derivatives $f'(x), f''(x), \ldots, f^{(\nu)}(x), \ldots$ have the property:*

$$\text{if } \xi \text{ is real and } f^{(\nu)}(\xi) = 0, \text{ then } f^{(\nu-1)}(\xi) f^{(\nu+1)}(\xi) < 0. \tag{2}$$

The rule is easily established now by the theorems of Budan-Fourier or Sturm; but its discovery was prior to that of these more general and better-known theorems, and was an important step towards them. It is due to de Gua.†

The function (1) satisfies the differential equation

$$f(x) + f'(x) + xf''(x) = 0; \tag{3}$$

from which we deduce, by repeated differentiation,

$$f^{(\nu-1)}(x) + \nu f^{(\nu)}(x) + xf^{(\nu+1)}(x) = 0. \tag{4}$$

The series (1) shows that $f^{(\nu)}(\xi)$ cannot vanish for $\xi \leqslant 0$. If $\xi > 0$ and $f^{(\nu)}(\xi) = 0$, then $f^{(\nu-1)}(\xi)$ and $f^{(\nu+1)}(\xi)$ have opposite signs on account

† *Histoire de l'Académie des Sciences*, 1741, p. 459.

of (4); i.e. (2) is true. (Two successive derivatives cannot vanish simultaneously, since then, by (4), all would vanish, and $f(x)$ would vanish identically.) Fourier applies de Gua's algebraical rule to $f(x)$ without further inquiry, and concludes that $f(x)$ has real zeros only.†

Cauchy expressed doubt about the validity of Fourier's reasoning.‡ Poisson went further, and refuted it by an example.§ The differential equation

$$af(x)+bf'(x)+cf''(x)=0 \tag{5}$$

may be treated like (3), if a, b, c are real and $ac>0$. We have in fact

$$af^{(\nu-1)}(x)+bf^{(\nu)}(x)+cf^{(\nu+1)}(x)=0,$$

and (2) follows. But the function

$$f(x)=e^x-e^{2x}=e^x(1-e^x) \tag{6}$$

is an integral of (5), with $a=2$, $b=-3$, $c=1$, and Fourier's argument shows that all the zeros of (6) are real, which is obviously false.

Poisson returned to the question on other occasions. One remark is worth quoting: 'Ces règles [the reference is to de Gua's] supposent qu'en différentiant un nombre de fois suffisant l'équation qu'on considère, on parvient enfin à une autre équation dont on sait que toutes les racines sont réelles.'‖ It is worth noticing that (as appears still more clearly from the context) Poisson is thinking here not only of polynomials but also of transcendental functions whose derivatives of high enough order have real zeros only; he regards the application of the rule of de Gua as legitimate for such transcendental functions, and for such only.††

Fourier defended himself brilliantly and energetically;‡‡ whether

† *Théorie de la chaleur*, Paris 1822, article 308, or *Œuvres*, edited by G. Darboux, Paris 1890, vol. 1, pp. 335–7.

‡ 'Quant à l'idée qu'a eue le même géomètre d'appliquer aux équations transcendantes les règles établies pour les équations algébriques, elle donne naissance à plusieurs difficultés' (*Œuvres complètes de Cauchy*, 2me série, vol. 2, p. 400).

§ *Journal de l'École Polytechnique*, vol. 12, 19. cahier, pp. 381–3. I present the example in the manner of G. Darboux (*Œuvres de Fourier*, vol. 1, p. 336) and with the numerical specialization considered by Fourier (*Œuvres*, vol. 2, p. 188).

‖ *Mémoires de l'Académie des Sciences*, vol. 8, pp. 367–8.

†† He changed his opinion afterwards concerning the applicability of the rule to (1). Compare *Mémoires de l'Académie des Sciences*, vol. 8, pp. 367–8, and vol. 9, p. 95.

‡‡ *Œuvres*, vol. 2, pp. 176–8, and 185–210. He objects to Poisson's example (6) that it does not satisfy (2) for $\xi=-\infty$ (but, in proving the result for (1),

correctly or not, we cannot tell even now. But he shifted his position more than he seems to admit, and especially on two points.

He gives a new proof† of the reality of all zeros of the function (1), which is entirely rigorous if we appeal to auxiliary lemmas which may be regarded as evident now (though hardly then). He proves first the reality of the zeros of a polynomial $f_n(x)$ of degree n, using a differential equation like (3), and the rule of de Gua, then of course valid. He then proves that $f_n(x) \to f(x)$ when $n \to \infty$.‡

Further, Fourier gives *conditions* for the validity of de Gua's rule. He also states a more general theorem which it is worth quoting verbally:§ 'Étant proposée une équation algébrique ou transcendante $\phi x = 0$ formée d'un nombre fini ou infini de facteurs réels ou imaginaires

$$\left(1-\frac{x}{\alpha}\right), \left(1-\frac{x}{\beta}\right), \left(1-\frac{x}{\gamma}\right), \left(1-\frac{x}{\delta}\right), \ldots,$$

... l'équation $\phi x = 0$ a précisément autant de racines imaginaires qu'il y a de valeurs réelles de x qui, substituées dans une fonction dérivée intermédiaire d'un ordre quelconque, rendent cette fonction nulle, et donnent deux résultats de même signe pour la fonction dérivée qui la précède et pour celle qui la suit. Par conséquent si l'on parvient à prouver qu'il n'y aucune valeur réelle de x qui, en faisant évanouir une fonction dérivée intermédiaire, donne le même signe à celle qui la précède, et à celle qui la suit, on est assuré que la proposée ne peut avoir aucune racine imaginaire.'

No proof of this theorem (by Fourier or another mathematician) is known. Stern‖ gave an invalid proof, and repeated in greater detail some affirmations of Fourier. Since then the question seems to have been neglected.

he did not examine $\xi = \pm\infty$, nor has the sign of the function for $\xi = \pm\infty$ or that of its derivatives anything to do with the criterion if it is stated correctly).

† *Œuvres*, vol. 2, pp. 200–4.

‡ After having completed his proof, Fourier tries to establish that the function (1) is a product of linear factors (*Œuvres*, vol. 2, pp. 204–7). Laguerre (see his *Œuvres*, Paris 1898, vol. 1, pp. 202–4), by combining a general theorem of his own with the differential equation (3), showed that the limiting process of Fourier establishes the decomposition into linear factors, i.e. that, in Laguerre's terminology generally adopted to-day, (1) is of genus 0. Laguerre's work reads quite like an analysis and completion of Fourier's proof, but he does not quote Fourier.

§ Fourier, *Analyse des équations déterminées*, Paris 1831, p. 66.

‖ *Crelle's Journal für die reine u. angew. Mathematik* vol. 22 (1841), pp. 1–62. See in particular pp. 49–50.

2. **Hypothetical theorems.** Before going further I have to explain a terminology often used in the sequel.

An integral function which takes real values on the real axis will be called a *real* integral function.

Let $g(z)$ be a real polynomial or a real integral function of genus 0 or 1, and γ a non-negative real constant. The integral function

$$f(z) = e^{-\gamma z^2} g(z)$$

will be called of *genus* 1*. Such a function is certainly real; its genus may be 0 or 1 or 2, but in the last case it is only 'slightly' higher than genus 1.

Let us suppose for a moment that neither the real integral function $f(z)$ nor any of its derivatives $f'(z), f''(z), \ldots$ possesses multiple zeros. Draw the curve whose equation in rectangular co-ordinates is

$$y = f^{(n-1)}(x),$$

and consider its positive minima and negative maxima, i.e. its extrema which are convex towards the x-axis; the abscissae of these points are called by Fourier the *critical points* of $f(z)$. In other terms, ξ is a critical point of $f(z)$, if

$$f^{(n)}(\xi) = 0, \qquad f^{(n-1)}(\xi)\, f^{(n+1)}(\xi) > 0.$$

We have to consider $n = 1, 2, 3, \ldots$, and each abscissa is counted as many times as there are different graphs in which it arises. The definition is more elaborate if there are multiple zeros (see section 5).

In this terminology an essential part of Fourier's theorem quoted in section 1 is embodied in the following proposition:

A. *A real integral function of genus* 0 *has just as many critical points as couples of imaginary zeros.*

This statement is narrower than that of Fourier's; it includes only the case where the product considered by Fourier is absolutely convergent, and does not apply, say, to $\sin z$.† If it could be proved, Fourier's argument about (1) would be completely justified, since this function is of order $\frac{1}{2}$ and thus certainly of genus zero, and fulfils the condition (2) of the rule of de Gua, i.e. has no critical points. Proposition A includes the assertion that the number of critical points and that of imaginary zeros are simultaneously infinite, and there lies the chief difference between it and the following proposition B.

Algebraic rules about the reality of roots extend naturally and

† Stern applied Fourier's argument to $\sin z$ (l.c., pp. 50–1). Poisson was, at a certain moment, inclined to believe that it applied to (1) but not to $\sin z$; see *Mémoires de l'Académie*, vol. 8 (1829), pp. 367–8.

often quite easily to integral functions of genus 1* having only a finite number of imaginary zeros, because these functions (and only these) may be approximated by real polynomials having a bounded number of non-real roots (see Lemma III in section 6). Thus we are led to the following proposition.

B. *If an integral function of genus* 1* *has only a finite number of imaginary zeros, it has just as many critical points as couples of imaginary zeros.*

There is no necessary contradiction between Poisson's statement about the zeros of derivatives quoted in the preceding section, and the proposition B. If we admit that they are simultaneously true we are led to the following proposition.

C. *If an integral function* $f(z)$ *of genus* 1* *has only a finite number of imaginary zeros, its derivatives from a certain one onward, let us say* $f^{(m)}(z)$, $f^{(m+1)}(z)$, ..., *have no imaginary zeros at all.*

3. **Theorems to prove.** I do not know whether any one of the theorems A, B, C is true or not. But some information about the question is contained in the following theorems I and II, which I shall prove in the sequel.

THEOREM I. *If an integral function* $f(z)$ *of genus* 1* *has only a finite number* J *of couples of imaginary zeros, then (as is well known) any derivative* $f^{(n)}(z)$ *of* $f(z)$ *has also only a finite number* J_n *of couples of imaginary zeros, and*

$$J \geqslant J_1 \geqslant J_2 \geqslant J_3 \geqslant \ldots.$$

Let K *be the number of critical points of* $f(z)$ *and*

$$J_\infty = \lim_{n \to \infty} J_n.$$

Then

$$J = K + J_\infty. \tag{7}$$

THEOREM II. *If the order of a real integral function* $f(z)$ *is less than* $\frac{2}{3}$ *and* $f(z)$ *has only a finite number of imaginary zeros, its derivatives from a certain one onward, let us say* $f^{(m)}(z)$, $f^{(m+1)}(z)$, ..., *have no imaginary zeros at all.*†

† I stated this theorem without proof in 1927 ('Ueber die algebraisch-funktionentheoretischen Untersuchungen von J. L. W. V. Jensen', *Matematisk-fysiske Meddelelser* (of the Danish Academy), vol. 7, No. 17, p. 27, footnote). Since the present paper went to press, M. Ålander published a proof and communicated a more general theorem stated by A. Wiman as probably true (*Opuscula Mathematica A. Wiman dedicata*, Lund 1930, pp. 79–98. p. 93). Ålander's proof is identical with my own which will be given in section 4, and the theorem suggested by Wiman is identical with the theorem C, suggested by Poisson's remark.

It follows from (7) that

$$K \leqslant J. \tag{8}$$

This inequality remains true for any function of genus 1* because it affirms nothing in the case $J = \infty$, excluded in Theorem I. The case of inequality for (8) presents itself in Poisson's example (6), where $K = 0$ and $J = \infty$. The function (6) is of genus and order 1, and thus it does not contradict the hypothetical theorem A, which affirms $K = J$ for the genus 0 and so for any order less than 1. Theorem I shows further that the hypothetical theorems B and C are equivalent, since, by (7), $J = K$ is equivalent to $J_\infty = 0$.

Theorem II shows that C and consequently B becomes certainly true under narrower conditions, namely, if we replace 'genus 1*' by 'order less than $\frac{2}{3}$'.

Theorems I and II permit us to draw a definite conclusion from Fourier's argument about (1). What Fourier actually proved is that $K = 0$ for the real integral function (1) whose order is $\frac{1}{2} < \frac{2}{3}$. If (1) had only a finite number J of couples of imaginary zeros, we could apply first Theorem I, then Theorem II, and obtain

$$J = J_\infty = 0.$$

That is, it follows from $K = 0$ that (1) has either an infinity of imaginary zeros or no such zeros.

4. **Proof of Theorem II.** I begin with the proof of Theorem II, which requires less algebraic machinery than that of Theorem I.

Let $\beta + i\gamma$, $\beta - i\gamma$, where β and γ are real, be a couple of conjugate imaginary zeros of the real integral function $f(z)$. Consider the ellipse whose minor axis has $\beta + i\gamma$ and $\beta - i\gamma$ as endpoints and whose major axis has the length $2\gamma\sqrt{n}$; it will be called a *Jensen ellipse of order* n of $f(z)$. A Jensen ellipse of the first order is a circle, a *Jensen circle* of $f(z)$. I have to use the following well-known lemma:†

LEMMA I. *If $f(z)$ is an integral function of genus 1*, the imaginary zeros of $f'(z)$ are contained in the Jensen circles of $f(z)$ and, more generally, the imaginary zeros of $f^{(n)}(z)$ in the Jensen ellipses of the n-th order of $f(z)$.*

Let the function considered in Theorem II be

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots. \tag{9}$$

Its order is less than $\frac{2}{3}$, and therefore its genus is 0, and because it

† J. L. W. V. Jensen, *Acta Mathematica*, vol. 36 (1913), pp. 181–95, see p. 190; J. L. Walsh, *Annals of Mathematics*, 2nd series, vol. 22 (1920), pp. 128–44; J. v. Sz. Nagy, *Jahresber. d. Deutsch. Math. Ver.*, vol. 31 (1922), pp. 238–51.

is real it is certainly contained in the class of functions of genus 1* to which Lemma I applies. $f(z)$ has only a finite number of imaginary zeros; thus its Jensen ellipses of order m cover only a finite region, interior to a circle

$$|z| \leqslant b+c\sqrt{m}, \tag{10}$$

where b and c are independent of m; they are determined by the situation of the imaginary zeros $\beta+i\gamma$ of $f(z)$, $b = \mathrm{Max}|\beta|$, $c = \mathrm{Max}|\gamma|$. The imaginary zeros of $f^{(m)}(z)$, if any, are, by Lemma I, situated in the circle (10).

On the other hand,† since the order of (9) is less than $\frac{2}{3}$, there is a constant α,

$$0 < \alpha < \tfrac{2}{3}, \tag{11}$$

such that

$$\lim_{n\to\infty} c_n n^{n/\alpha} = 0. \tag{12}$$

It follows from (11), by a familiar remark,‡ that there are arbitrarily large m such that

$$|c_m| m^{m/\alpha} > |c_{m+k}|(m+k)^{(m+k)/\alpha}$$

for $k = 1, 2, 3, \ldots$. Therefore

$$\left|\frac{c_{m+k}}{c_m}\right| < \frac{m^{m/\alpha}}{(m+k)^{(m+k)/\alpha}} < \frac{1}{(m+k)^{k/\alpha}},$$

$$\left|\frac{f^{(m)}(z)}{m!\,c_m}\right| = \left|\sum_{k=0}^{\infty} \frac{(m+k)!}{m!\,k!}\frac{c_{m+k}}{c_m} z^k\right|$$

$$\geqslant 1 - \sum_{k=1}^{\infty} \frac{(m+1)(m+2)\ldots(m+k)}{k!}\left|\frac{c_{m+k}}{c_m}\right| |z|^k$$

$$> 1 - \sum_{k=1}^{\infty} \frac{(m+k)^k}{k!}\frac{1}{(m+k)^{k/\alpha}} |z|^k$$

$$> 1 - \sum_{k=1}^{\infty} \frac{|z|^k m^{k-k/\alpha}}{k!} = 2 - \exp(|z| m^{1-1/\alpha}).$$

† The following argument and the result concerning (13) are due to M. Ålander, 'Sur le déplacement des zéros des fonctions entières par leur dérivation' (*Thèse*, Upsala, 1914, pp. 6–8).

‡ See e.g. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze* (Berlin 1925), vol. 1, p. 8, no. 108.

Therefore $f^{(m)}(z)$ has all its zeros outside a circle, namely, in the region

$$|z| > m^{\frac{1}{\alpha}-1} \log 2. \tag{13}$$

Now, by (11),

$$\frac{1}{2} = \frac{3}{2} - 1 < \frac{1}{\alpha} - 1;$$

and thus, if m is sufficiently great, the regions (10) and (13) have no common point and $f^{(m)}(z)$ has no imaginary zero. It follows, as is well known,† that also $f^{(m+1)}(z)$, $f^{(m+2)}(z)$, ... have only real zeros.

5. **Critical points.** Suppose that $f(x)$ takes real values and is a regular analytic function in some interval of the real axis. Suppose that ξ is a real zero of $f'(x)$ of multiplicity m but not a zero of $f(x)$. That is

$$f(\xi) \neq 0, f'(\xi) = f''(\xi) = \ldots = f^{(m)}(\xi) = 0, f^{(m+1)}(\xi) \neq 0. \tag{14}$$

Put

(i) $k = \dfrac{m}{2}$, if m is even,

(ii α) $k = \dfrac{m+1}{2}$, if m is odd and $f(\xi) f^{(m+1)}(\xi) > 0$,

(ii β) $k = \dfrac{m-1}{2}$, if m is odd and $f(\xi) f^{(m+1)}(\xi) < 0$.

If $k > 0$ we shall say that ξ is a *critical zero of* $f'(x)$ *of the* (*critical*) *multiplicity* k.

Observe that:

(a) If $m = 1$, the point ξ is or is not a critical zero of $f'(x)$ according as $f(\xi) f''(\xi)$ is positive or negative, that is, according as the convex or the concave side of the curve $y = f(x)$ is turned towards the x-axis at the point ξ, where $f(x)$ reaches an extremum. If $m = k = 1$, the point ξ is simple (has the multiplicity 1) as a zero and also as a critical zero. A zero of $f'(x)$ of multiplicity m and of critical multiplicity k may be interpreted as arising from m coincident simple real zeros, k of which are critical and $m-k$ non-critical.

(b) Critical zeros of $f'(x)$ may arise also from coincident imaginary zeros. Assume that the analytic functions $f_1(x)$, $f_2(x)$, ... take real values along an interval of the real axis, and tend towards $f(x)$, uniformly in a rectangle of the complex plane that contains the interval in question in its interior. Consider the zero ξ of $f'(x)$, of

† E. Laguerre, *Œuvres*, vol. 1, pp. 178–80.

multiplicity m and critical multiplicity k, and a given small neighbourhood of ξ in the real points of which $f(x)f^{(m+1)}(x)$ keeps a constant sign. For sufficiently great n, the sign of $f_n(x)f_n^{(m+1)}(x)$ will be the same, and the neighbourhood will contain just m zeros of $f_n'(x)$ among which $2j^*$ may be imaginary and k^* critical. If we examine the cases (i), (ii α), (ii β) we see easily that in each of them

$$2k = 2j^* + 2k^*.$$

(*c*) If x passes through ξ, the sequence $f(x), f'(x), f''(x), \ldots$ considered in the Budan-Fourier rule,† loses $2k$ variations owing to the critical zero ξ of $f'(x)$ of multiplicity k, i.e. it loses just $2k$ variations if there is no derivative of $f(x)$, beyond those mentioned in (14), that vanishes at the point ξ. These $2k$ variations are 'wasted' for $f(x)$, i.e. they indicate no root of $f(x)$, but they are not 'wasted' for $f'(x)$, i.e. the shorter sequence $f'(x), f''(x), \ldots$ loses just as many variations as there are roots of $f(x)$ in the point ξ.

A critical zero of $f^{(n)}(x)$, $n \geqslant 1$, is related in just the same way to the curve $y = f^{(n-1)}(x)$ as a critical zero of $f'(x)$ to the curve $y = f(x)$. All the critical zeros of $f'(x), f''(x), f'''(x), \ldots$ are called *critical points* of $f(x)$. The multiplicity of a critical point of $f(x)$ is the sum of the (critical) multiplicities of the critical zeros of $f'(x), f''(x), \ldots$ lying in that point. If c is the multiplicity of a critical point ξ of $f(x)$, then just $2c$ variations of the sequence $f(x), f'(x), f''(x), \ldots$ are 'wasted' for $f(x)$ when x passes through ξ.

Consider an interval a, b. Suppose that, multiple zeros of any kind being counted with the proper multiplicity,

N is the number of zeros of $f(x)$ in a, b;

N_1 is the number of zeros of $f'(x)$ in a, b;

K_ν is the number of critical zeros of $f^{(\nu)}(x)$ in a, b $(\nu = 1, 2, 3, \ldots)$.

Suppose now that both at a and at b the function $f(x)$ and its derivatives are different from zero, and further that a certain derivative $f^{(l)}(x)$ is different from zero throughout a, b. Let $\{f(x)\}$ be the number of variations in the sequence $f(x), f'(x), f''(x), \ldots f^{(l)}(x)$, and $\{f'(x)\}$ that in the shorter sequence $f'(x), f''(x), \ldots f^{(l)}(x)$.

Then, by (*c*) and the Budan-Fourier rule,

$$N + 2(K_1 + K_2 + \ldots + K_{l-1}) = \{f(a)\} - \{f(b)\}, \tag{15}$$

$$N_1 + 2(K_2 + \ldots + K_{l-1}) = \{f'(a)\} - \{f'(b)\}. \tag{16}$$

† The Budan-Fourier rule in the generality needed here was discussed by A. Hurwitz, *Math. Annalen*, vol. 71 (1912), pp. 584–91.

Since $$\{f(a)\}-\{f'(a)\} = \tfrac{1}{2}[1-sgf(a)f'(a)],$$

and there is an analogous relation for b, we infer from (15) and (16) that

$$N+2K_1-N_1 = \tfrac{1}{2}[sgf(b)f'(b)-sgf(a)f'(a)]. \tag{17}$$

Though the proof supposed more, (17) is valid on the sole hypothesis that $f(a)f(b)f'(a)f'(b) \neq 0$. This is easy to confirm by direct geometrical considerations if we divide a, b into partial intervals by the zeros of $f(x)$ and $f'(x)$. The following special or rather limiting cases are particularly easy.

(*d*) If the interval is *closed*, and both its endpoints are roots of $f(x)$, then

$$N_1 = N-1+2K_1. \tag{18}$$

(*e*) If $f(x)$ is a polynomial, the interval is *open*, and one of its extremities is a root of $f(x)$ and the other $+\infty$ or $-\infty$, then

$$N_1 = N+2K_1. \tag{19}$$

6. **Integral functions of genus 1*.** An integral function $f(z)$ of genus 1* has the form

$$f(z) = \alpha e^{-\gamma z^2+\beta z} z^q \Pi\left(1-\frac{z}{\alpha_\mu}\right) e^{\frac{z}{\alpha_\mu}} \Pi\left(1-\frac{z}{a_\nu}\right)\left(1-\frac{z}{\bar{a}_\nu}\right) e^{\frac{z}{a_\nu}+\frac{z}{\bar{a}_\nu}}, \tag{20}$$

where

$$a_\nu = \beta_\nu+i\gamma_\nu, \qquad \bar{a}_\nu = \beta_\nu-i\gamma_\nu,$$

α, β, γ, α_μ, β_ν, γ_ν are real, $\alpha \neq 0$, $\gamma \geqq 0$, $\alpha_\mu \neq 0$, $\gamma_\nu > 0$, q is a non-negative integer, and each of the two products Π is either absolutely convergent, or has a finite number of factors, or has no factors at all (in which case it is to be replaced by 1). I prove here two lemmas, one on the critical points of integral functions of genus 1* and one on their approximation by polynomials.

LEMMA II. *If $f(z)$ is an integral function of genus* 1*, *the critical zeros of $f'(z)$ are contained in the Jensen circles of $f(z)$ and, more generally, the critical zeros of $f^{(n)}(z)$ in the Jensen ellipses of order n of $f(z)$.*

I will prove only the first part of the lemma, that about the Jensen circles. The second part, which by the way will not be used in the sequel, follows from the first and from Lemma I by an easy geometrical argument† also used in the proof of Lemma I.

† See J. v. Sz. Nagy, *Jahresber. d. Deutsch. Math. Ver.*, vol. 31 (1922), pp. 243–4.

Differentiating $\log f(z)$ twice we obtain

$$\frac{f'(z)^2-f(z)f''(z)}{f(z)^2} =$$

$$= 2\gamma + \frac{q}{z^2} + \sum \frac{1}{(z-\alpha_\mu)^2} + 2\sum \frac{(z-\beta_\nu)^2-\gamma_\nu{}^2}{[(z-\beta_\nu)^2+\gamma_\nu{}^2]^2}. \quad (21)$$

Exclude the case (in which there is nothing to prove) that (21) vanishes identically, or what is the same, that $f(z)$ is of the form $\alpha e^{\beta z}$. If ξ is a real zero of $f'(x)$ but not a zero of $f(x)$, and ξ is exterior to all the Jensen circles, that is if

$$(\xi-\beta_\nu)^2 > \gamma_\nu{}^2,$$

then we obtain from (21)

$$-f(\xi)f''(\xi) > 0.$$

Thus ξ is a simple and non-critical zero of $f'(z)$, which is the relevant part of Lemma II.

Lemma II expresses a relation between the situation of the imaginary zeros and that of the critical points of an integral function of genus 1*. It is *a fortiori* valid for polynomials; and this special case may be part of an answer to a rather obscure question of Gauss.† The geometrical relation given by Lemma II may be expressed in a different form:

Let $f(z)$ be an integral function of genus 1 and ξ a critical zero of $f'(z)$. Let two straight lines passing through ξ divide the complex plane into four right angles two of which are bisected by the real axis. Then the two right angles which are not bisected by the real axis contain each at least one complex zero of $f(z)$. If ξ were a critical zero of $f^{(n)}(z)$, instead of being one of $f'(z)$, then, in order to obtain the same conclusion, we must open the two angles just considered till the aperture of each becomes* $2\arctan\sqrt{n}$.

LEMMA III. *If the integral function $f(z)$ of genus 1* has only a finite number of imaginary zeros, all situated in a certain closed domain R of the complex plane, then there is a sequence of real polynomials tending to $f(z)$, uniformly in R, and such that each polynomial of the sequence has just the same zeros in R as $f(z)$ and only real zeros outside R.*

† *Werke*, vol. 3, p. 120. The question is whether there is a 'definite' one-to-one relation between critical points and couples of imaginary zeros: 'ob jeder *bestimmten* Lücke ein *bestimmtes* Paar imaginärer Wurzeln angehört.' Did Gauss mean a geometrical relation?

If $f(z)$ has the form (20), polynomials of the required kind are given by the formula

$$\alpha\left(1-\frac{\gamma z^2}{P}\right)^P z^q\left(1+z\frac{\beta+\Sigma_p\,\alpha_\mu^{-1}+\Sigma\,(a_\nu^{-1}+\bar{a}_\nu^{-1})}{P}\right)^P\times$$

$$\times\Pi_p\left(1-\frac{z}{\alpha_\mu}\right)\Pi\left(1-\frac{z}{a_\nu}\right)\left(1-\frac{z}{\bar{a}_\nu}\right);$$

here p runs through all integers from a certain one onward, the product Π_p and the sum Σ_p are extended over the α_μ whose modulus does not exceed p, and $P=P(p)$ is a suitably chosen (large) integer.†

7. **Proof of Theorem I.** The main point of Theorem I is contained in the following:

LEMMA IV. *If $f(z)$ is an integral function of genus 1* and has $2J$ imaginary zeros, and its derivative $f'(z)$ has $2J_1$ imaginary and K_1 critical zeros, then*

$$J=J_1+K_1. \tag{22}$$

Relation (22) is easy to prove for real polynomials, and the following proof extends it with the help of approximating polynomials (Lemma III) to functions of genus 1*; this process succeeds because the imaginary and critical zeros of the derivative cannot 'escape to infinity' (Lemmas I and II). But the details need very careful handling.

Consider a rectangle R with horizontal and vertical sides, symmetrical with respect to the real axis. Choose R so that

(i) R contains in its interior all the $2J$ couples of imaginary roots of $f(z)$ with the corresponding Jensen circles, and both its vertical sides are halved by a root of $f(z)$. These roots are real and will be called a and b.

The choice of R to satisfy the second condition in (i) is certainly possible if $f(z)$ possesses an *unlimited number of positive and negative zeros*. I assume this in the following proof, which requires a slight modification—use of the more complicated (17) instead of (18) and (19)—in the other cases where $f(z)$ possesses only a finite number of zeros of one sign or altogether.

It follows from (i), by Lemmas I and II, that the $2J_1$ imaginary and the K_1 critical zeros of $f'(z)$ are contained in the interior of R.

† Compare Hilfssatz ii, p. 20, (*Matematisk-fysiske Meddelelser*, vol. 7, No. 17). Note that 'erhöhtes Genus 1' and 'genus 1*' mean the same thing.

(ii) Let $f_n(z)$ be a real polynomial of degree n which has just the same zeros in the closed rectangle R as $f(z)$, and no imaginary zeros outside R.

Let $f(z)$, and therefore $f_n(z)$, have N real zeros in the closed rectangle R. If the number of zeros of $f_n(z)$ outside R (all of which are real) is n', then

$$n = 2J+N+n'. \tag{23}$$

(iii) Let the zeros of $f'(z)$ inside R be the centres of small circles all interior to R and exterior to each other. Choose $f_n(z)$, according to Lemma III, so that $f_n'(z)$ has in each of these small circles exactly the same number of roots as $f'(z)$ and no other roots inside or on the boundary of R, except possibly roots in a and b if these points are multiple zeros of $f(z)$. This choice is possible, by Lemma III and well-known general theorems.

By (i) and (ii), $f(z)$ and $f_n(z)$ have the same imaginary roots and Jensen circles, all inside R. Therefore, by Lemmas I and II, all imaginary and critical zeros of $f'(z)$ and $f_n'(z)$ are contained in the interior of R. Let $f_n'(z)$ have $2J_1^*$ imaginary, K_1^* critical, and N_1^* real zeros in the closed rectangle R, the corresponding numbers for $f'(z)$ being $2J_1$, K_1, and N_1.

Since $f_n'(z)$ has no critical zeros outside R, it has, by (e) of section 5, just the same number of real zeros outside R as $f_n(z)$. Therefore, compare (23),

$$n-1 = 2J_1^*+N_1^*+n'. \tag{24}$$

By (d) of section 5 we have

$$N_1^* = N-1+2K_1^*. \tag{25}$$

By (iii) and (b) of section 5, we have

$$2J_1^*+2K_1^* = 2J_1+2K_1. \tag{26}$$

Combining (23) and (24) we obtain

$$N_1^* = N-1+2J-2J_1^*.$$

This gives, compared with (25),

$$J = J_1^*+K_1^* = J_1+K_1,$$

by (26). Thus (22) is proved.

Suppose that $f^{(\nu)}(z)$ has $2J_\nu$ imaginary and K_ν critical zeros, ($\nu = 1, 2, 3, \ldots$). Since $f(z)$ is of genus 1* and has only a finite num-

ber of imaginary zeros, $f'(z), f''(z), \ldots$ are also† of genus 1*. Thus we have a system of equations like (22), viz.

$$J_1 = J_2 + K_2,\ J_2 = J_3 + K_3, \ldots;$$

and thus

$$J = J_n + K_1 + K_2 + \ldots + K_n,$$

$$J = J_\infty + K_1 + K_2 + K_3 + \ldots$$

which is Theorem I.

† This is an easy development of a theorem of Laguerre (*Œuvres*, vol. 1, pp. 178–80). See e.g. G. Pólya, *Crelle's Journal für die reine u. ang. Math.*, vol. 145 (1914), pp. 224–9, p. 245, Theorem II, from which it follows at once.

ELECTRICAL NOTES

By F. B. PIDDUCK

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I. Relation of two-dimensional problems in electrostatics and electromagnetism

THE evident fact that the electric potential V of an infinite line charge and the magnetic potential Ω of an infinite straight current are conjugate functions can be generalized to dielectrics and magnetic media. We shall pass briefly over the direct proof since the theorem is easy to find.

Suppose for simplicity of enunciation that an infinite cylinder S of any cross-section separates in one problem two dielectrics of dielectric constants K_1 and K_2, and in the other two magnetic media of permeabilities μ_1 and μ_2. Let there be charges of e_1 and e_2 electrostatic units per unit length in two lines parallel to the generators in the first and second dielectrics respectively, and currents i_1 and i_2 in the same lines in the second problem. Let V be the electric potential in the first problem and U its conjugate function such that

$$V+iU = f(x+iy).$$

The conditions of each problem are five: Laplace's equation in two dimensions, two conditions expressing the behaviour of the functions infinitely near the two lines, and two boundary conditions on S. The relations $\partial V/\partial n = \partial U/\partial s$, $\partial V/\partial s = -\partial U/\partial n$ are just such as will transform a tangential boundary condition into a normal one, and conversely, and a simple calculation shows that all the conditions are satisfied if $\Omega = \lambda_1 U$ in the first space and $\lambda_2 U$ in the second space, where

$$K_1\mu_1 = K_2\mu_2, \qquad \lambda_1\mu_1 = \lambda_2\mu_2, \qquad e_1/i_1 = e_2/i_2.$$

The lines of magnetic force in the second problem coincide with the equipotential curves in the first, and if one problem is solved by images the other has the same image system.*

The constancy of $K\mu$ suggests a more comprehensive view. The conjugate function relation is familiar in Rayleigh's theory of electric waves on cylindrical conductors.† Can we generalize it to cover a

* Cf. G. F. C. Searle, *Electrician*, 28 Jan. 1898.

† Rayleigh, *Phil. Mag.* [4], 44, p. 199 (1897); *Scientific Papers*, 4, p. 327.

dielectric constant which varies from point to point of the cross-section? Obviously only if the permeability varies inversely so as to produce a wave velocity the same at all points, so that the wave fronts remain plane. Consider the electric waves on a number of conducting cylinders parallel to the axis of z, the dielectric constant K and permeability μ being functions of x, y, subject to $K\mu$ having the constant value c^2/v^2. Then the equations

$$K\partial\mathbf{E}/\partial t = c\,\mathrm{curl}\,\mathbf{H},\ \mathrm{div}\,K\mathbf{E} = 0,\ -\mu\partial\mathbf{H}/\partial t = c\,\mathrm{curl}\,\mathbf{E},\ \mathrm{div}\,\mu\mathbf{H} = 0$$

admit the solution

$$E_x = -\frac{\partial V}{\partial x},\quad E_y = -\frac{\partial V}{\partial y},\quad E_z = 0,$$

$$H_x = \frac{c}{\mu v}\frac{\partial V}{\partial y} = \frac{Kv}{c}\frac{\partial V}{\partial y},\quad H_y = -\frac{c}{\mu v}\frac{\partial V}{\partial x} = -\frac{Kv}{c}\frac{\partial V}{\partial x},\quad H_z = 0,$$

where V is a function of x, y and $z-vt$ satisfying the equation

$$\frac{\partial}{\partial x}\left(K\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial V}{\partial y}\right)=0.$$

Evidently a solution of the electrostatic problem suffices to solve the wave problem, and in particular the problem of the field of high-frequency currents along the cylinders considered in the electrostatic problem. If we make the cylinders infinitely small in cross-section, as well as the wave-length infinitely long, we have the analogy with which we began.

II. The electric oscillations on a thin terminated wire.

The interest of Abraham's theory* of the electric oscillations on a prolate spheroid being in its application to a thin rod, one could wish it brought into agreement with Pocklington's theory† of the oscillations on a thin wire of any form. Rayleigh‡ remarked that the limit of a spheroid is not a uniform rod but one which tapers according to a definite law; but gave reasons, from the solution of another problem, for supposing that the magnitude of the cross-section is immaterial provided it is small. The difficulty itself is, however, only apparent in our problem. All we require are solutions which make the electric force intersect the wire at right angles in the limit, and this is provided by Abraham's solutions, or by the

* M. Abraham, *Wied. Ann.* 66, 435 (1898).
† H. C. Pocklington, *Proc. Camb. Phil. Soc.* 9, 329 (1897).
‡ *Phil. Mag.* [6] 8, 105 (1904); *Scientific Papers*, 5, p. 198.

oscillations on any other surface which can be made to tend to a thin terminated wire.

Pocklington's main result for a thin wire is that the distribution of current is simple harmonic in the arc s, provided that the wire is either closed or has its extremities at infinity. The restriction is unnecessary. Pocklington carried out an integration by parts in which he replaced $-\int \lambda \, d\Pi$ by $\int \Pi \, d\lambda$, the distribution of Hertzian oscillators along the wire being proportional to a function λ, simple harmonic in t. We may remark parenthetically that the sign of Pocklington's coefficient α should be changed in order to make it clear that we consider only waves propagated outwards from the elementary oscillators. Larmor* observed that, since the current in the wire is proportional to $\partial\lambda/\partial t$, λ vanishes at a free end. Thus Pocklington's conclusions apply equally to a terminated wire. Since Abraham found that the distribution of current is simple harmonic in s for the limit of a thin spheroid, there is complete accord, and both make the fundamental wave-length twice the length of the rod. The result is now seen to hold for a rod bent into any curve.

* J. Larmor, *Nature*, 67, 361 (1903).

THE ZEROS OF CERTAIN INTEGRAL FUNCTIONS

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Introduction.

1.1. I propose to consider functions of the form

$$f(z) = f(x+iy) = f(re^{i\theta}) = \int_a^b e^{zt}\phi(t)\,dt, \tag{1.11}$$

where $\phi(t)$ is a complex function, integrable in the sense of Lebesgue. The functions

$$U(z) = \int_a^b \cos zt\, u(t)\,dt, \quad V(z) = \int_a^b \sin zt\, v(t)\,dt \tag{1.12}$$

may be reduced to the form (1.11) by simple transformations. Among them are many well-known functions such as Bessel functions.

The problem here is to determine approximately the total number of zeros, and the position of the zeros for which r is large. Titchmarsh* has proved a number of very general theorems of this character. In these it is supposed (as is obviously permissible without real loss of generality) that a and b are the effective limits of integration of $\phi(t)$, that is to say that

$$\int_a^{a+\delta} |\phi(t)|\,dt > 0, \quad \int_{b-\delta}^b |\phi(t)|\,dt > 0, \tag{1.13}$$

for every $\delta > 0$. It is also supposed that $f(0) \neq 0$. This hypothesis also may be made without loss of generality, since $f(0) = 0$ involves

$$f(z) = -z\int_a^b e^{zt}\psi(t)\,dt = -z\,g(z),$$

where $\psi(t) = \int_a^t \phi(u)\,du$, and $g(z)$ is a function of the same type as $f(z)$ with a zero at the origin of smaller order.

Titchmarsh's chief results are as follows:

THEOREM I. *If $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}, \ldots$, where $0 < r_1 \leqslant r_2 \leqslant \ldots$, are the zeros of $f(z)$, then the series*

$$\sum_1^\infty \frac{\cos\theta_n}{r_n}$$

is absolutely convergent.

* E. C. Titchmarsh, *Proc. London Math. Soc.* (2) 25 (1926), 283–302.

THEOREM II. *If $n(\rho)$ is the number of zeros of $f(z)$ for which $r \leqslant \rho$, then*

$$n(\rho) \sim \frac{b-a}{\pi}\rho$$

when $\rho \to \infty$.

THEOREM III. *The series*

$$\sum_1^\infty \frac{\sin\theta_n}{r_n}$$

is conditionally convergent, and we can write

$$f(z) = f(0)e^{\frac{1}{2}(a+b)z}\prod_1^\infty\left(1-\frac{z}{z_n}\right),$$

the product being conditionally convergent.

1.2. Pólya* has proved a number of theorems of a different type concerning the zeros of the functions (1.11). In Pólya's theorems the hypotheses are much more restrictive, and the conclusions much more precise. For example, he proves

THEOREM IV. *If $\phi(t)$ is positive and non-decreasing, then all the zeros of $f(z)$ lie in the closed half-plane $x \leqslant 0$.*

THEOREM V. *If $\phi(t)$ is continuous, positive and differentiable, except at a finite number of points, and if*

$$\alpha \leqslant \frac{\phi'(t)}{\phi(t)} \leqslant \beta \qquad (a < t < b),$$

then all the zeros of $f(z)$ lie in the strip $\alpha \leqslant x \leqslant \beta$.

He also proves very precise results concerning the reality and location of the zeros of the functions (1.12); thus if $v(t)$ is a positive increasing function, which is convex in the interval $(0, 1)$, and if $v(t+0) = 0$, then all the zeros of

$$V(z) = \int_0^1 \sin zt\, v(t)\, dt$$

are real, and there is exactly one zero inside each of the intervals $(\pi, \frac{3}{2}\pi)$ $(2\pi, \frac{5}{2}\pi)$.... The well-known results for Bessel functions are included among these as special cases.

* G. Pólya, *Math. Zeitschrift*, 2 (1918), 352–83.

Very precise results have been obtained by Hardy,* for particular functions, such as

$$f(z)=\int_0^1 e^{zt}t^{a-1}\left(\log\frac{1}{t}\right)^{s-1}dt=\sum_0^\infty \frac{\Gamma(s)z^n}{\Gamma(n+1)(a+n)^s}.$$

The first new theorem of this paper, viz. VI, and the substance of VII are also due to Professor Hardy. I should like to thank him for permission to use his manuscript, and for many valuable suggestions and criticisms.

1.3. My object here is to prove a number of theorems intermediate in generality between those of Titchmarsh and Hardy. These theorems are of a rather different character from most of Pólya's, more general in some respects and less in others. I begin with a few general remarks by way of explanation.

When $z\to\infty$ or $-\infty$ by real values, the behaviour of $f(z)$ is dominated by the behaviour of $\phi(t)$ near b or a, and this remains true, roughly, when $z\to\infty$ in the part of the plane well removed from the imaginary axis. It is essential, in order to obtain any lower limit for $n(\rho)$, to make some hypothesis which ensures that $\phi(t)$ shall not be 'too small' near a or b. In Titchmarsh's work, this hypothesis is (1.13). For my purposes a stronger hypothesis is required, and I suppose throughout the paper that $\phi(t)$ is continuous at a and b, and that neither $\phi(a)$ nor $\phi(b)$ is 0. This hypothesis ensures that the regions

$$|\cos\theta|\geqslant\epsilon>0,$$

at least, are free from zeros.

Near the imaginary axis $f(z)$ behaves very much like the Fourier coefficient of $\phi(t)$. It follows from what we have just said that the zeros must lie 'near the imaginary axis', so that it is not surprising to find that there is an analogy between the order of the Fourier coefficients of $\phi(t)$ and the distribution of the zeros of $f(z)$. The zeros are distributed more or less like those of the standard function,

$$\frac{1}{z}\sinh\left\{\tfrac{1}{2}(b-a)\left(z-\tfrac{1}{2}\log\frac{\phi(a)}{\phi(b)}\right)\right\};$$

the error in $n(\rho)$, in nearly all the cases considered, is ρ times the order of the $[\rho]$th Fourier coefficient. This rule also holds in the general case considered by Titchmarsh, since, by Theorem II, the error for

* G. H. Hardy, *Proc. London Math. Soc.* (2) (1904), 401–31.

$n(\rho)$ is $o(\rho)$, while the $[\rho]$th Fourier coefficient of $\phi(t)$ is $o(1)$; but I shall prove that there are exceptions.

There are, naturally, corresponding results if $\phi(t)$ tends to 0 or to ∞ at the ends in a fairly regular manner, but these I must leave for another time.

1.4. It is convenient to take -1 and 1 as the limits of integration. The general case can be reduced to this by an obvious linear transformation.

I shall also suppose that $\phi(1)=\phi(-1)=1$. The case in which $\phi(t)$ does not vanish at the ends can be reduced to this. For putting

$$z=z_1+\tfrac{1}{2}\log\frac{\phi(-1)}{\phi(1)}, \quad \phi_1(t)=\left\{\frac{\phi(-1)}{\phi(1)}\right\}^{\frac{1}{2}t}\frac{\phi(t)}{\{\phi(1)\phi(-1)\}^{\frac{1}{2}}},$$

and making the functions one-valued by convention, we have

$$f(z)=\{\phi(1)\phi(-1)\}^{\frac{1}{2}}\int_{-1}^{1}e^{z_1t}\phi_1(t)\,dt=\{\phi(1)\phi(-1)\}^{\frac{1}{2}}g(z_1)$$

where $\phi_1(1)=\phi_1(-1)=1$.

Throughout much of the working we have to consider

$$zf(z)=f_1(z)=Re^{i\Theta},$$

where $R=R(z)$, $\Theta=\Theta(z)$ are real, but $n(\rho)$ always denotes the number of zeros of $f(z)$ for which $r\leqslant\rho$, except in 3.21 and 3.22, where it obviously refers to the zeros of $F(z)$.

$\phi(t)$ an integral.

2.1. Hardy has proved

THEOREM VI. *If $\phi(t)$ is an integral, i.e. if $\phi(t)=\int_{t_0}^{t}\phi'(u)\,du$, then the zeros of $f(z)$ are determined asymptotically by the formula*

$$z=\pm m\pi i+\epsilon_m,$$

where m is a positive integer and $\epsilon_m\to 0$.

We shall require the following lemma, which has been proved by Titchmarsh.*

LEMMA A. *If $\psi(t)$ is integrable, then,*

$$\int_{-1}^{1}e^{zt}\psi(t)\,dt=o\{e^{r|\cos\theta|}\}$$

uniformly with respect to θ.

* Loc. cit. 288.

Integrating by parts,

$$f(z) = \frac{e^z - e^{-z}}{z} - \frac{1}{z}\int_{-1}^{1} e^{zt}\phi'(t)\,dt.$$

Since $\phi'(t)$ is integrable,

$$\begin{aligned} f_1(z) &= e^z - e^{-z} - o\{e^{r|\cos\theta|}\} \\ &= e^z(1+\epsilon_r) - e^{-z}(1+\epsilon_r), \end{aligned}$$

where ϵ_r denotes generally a function of r and θ which tends to 0 uniformly in θ, when $r \to \infty$. At a zero of $f(z)$

$$e^z(1+\epsilon_r) = e^{-z}(1+\epsilon_r),$$

$$e^{2z} = 1+\epsilon_r;$$

and hence

$$2z = \pm 2m\pi i + \epsilon_m,$$

where m is a positive integer and $\epsilon_m \to 0$.

It remains to prove that one and only one zero of $f(z)$ is associated with each large integral value of m. The function $\sinh z$ has exactly one zero at each point $z = \pm m\pi i$. Let δ be any sufficiently small positive number, and let us surround each of these points by a circle of radius δ. Then

$$|\sinh z| > \eta(\delta) > 0$$

on the circle. Also the ratio between the moduli of the two terms of $\sinh z$ lies between certain fixed limits, depending only on δ, on the circle. Hence, on the circle, we may write

$$f_1(z) = \sinh z\,(1+\epsilon_m),$$

where $\epsilon_m \to 0$ as $m \to \infty$. It follows that $f_1(z)$ has, for large m, the same number of zeros inside the circle as $\sinh z$, that is to say, one.

2.2. We have thus determined the large zeros of $f(z)$ with considerable precision. It is easy to show also that the total number of zeros is the same for $f_1(z)$ and $\sinh z$, i.e. that

$$n(\rho) = 2\left[\frac{\rho}{\pi}\right],$$

when ρ is large and ρ/π is not too near an integer.

$\phi(t)$ of bounded variation.

3.11. We know that, if a_m is the mth Fourier coefficient of $\phi(t)$, then

$$a_m = o(1) \text{ for an integrable } \phi,$$

$$a_m = o(m^{-1}) \text{ when } \phi \text{ is an integral.}$$

In either case the error in $n(\rho)$ is ρ times the order of $a_{[\rho]}$.

If ϕ is of bounded variation, $a_m = O(m^{-1})$, and if ϕ belongs to Lip α, i.e. if

$$\phi(t+h)-\phi(t) = O(|h|^{\alpha}),$$

then $a_m = O(m^{-\alpha})$. Analogy suggests that the corresponding errors in $n(\rho)$ should be $O(1)$ and $O(\rho^{1-\alpha})$. This is in fact true, but the proofs are decidedly more difficult.

Since a function of bounded variation is the difference of two monotonic functions, it is natural to expect also an analogue of Theorem IV. This is Theorem VII below (an essential preliminary to Theorem VIII, which is the theorem for $n(\rho)$ when ϕ is of bounded variation).

THEOREM VII. *If $\phi(t)$ is of bounded variation, then all the zeros of $f(z)$ lie in the strip $|x| < K$ for some K.*

THEOREM VIII. *If $\phi(t)$ is of bounded variation, then, as $\rho \to \infty$,*

$$n(\rho) = \frac{2\rho}{\pi} + O(1).$$

The following example shows that there is no one-to-one correspondence between the zeros of $f(z)$ and $\sinh z$ as in the 'integral' case. Suppose that, $f(z)$ not having been reduced to the standard case,

$$\begin{aligned} \phi(t) &= -1 \qquad (-1 < t < 0) \\ &= 1 \qquad (0 < t < 1). \end{aligned}$$

Then the zeros of $$f(z) = \frac{4\sinh^2 \tfrac{1}{2}z}{z^2}$$

occur in pairs at the points $z = \pm 2m\pi i$, $m = 1, 2, \ldots$, instead of singly at or near the points $z = \pm(m-\tfrac{1}{2})\pi i$, as they would if the result of Theorem VIII still held.

3.12. *Proof of Theorem VII.* Suppose first that $x \geqslant 0$. Integrating by parts

$$\begin{aligned} f(z) &= \frac{e^z - e^{-z}}{z} - \frac{1}{z}\int_{-1}^{1} e^{zt}\, d\phi \\ &= \frac{e^z - e^{-z}}{z} - \frac{1}{z}\int_{-1}^{1-\delta} e^{zt}\, d\phi - \frac{1}{z}\int_{1-\delta}^{1} e^{zt}\, d\phi. \end{aligned}$$

Now $$\left|\int_{-1}^{1-\delta} e^{zt}\, d\phi\right| \leqslant e^{x(1-\delta)} \int_{-1}^{1-\delta} |d\phi| < H e^{x(1-\delta)},$$

and $$\left|\int_{1-\delta}^{1} e^{zt}\, d\phi\right| < e^{x} \operatorname*{Max}_{1-\delta < t_1 < t_2 \leqslant 1} |\phi(t_2) - \phi(t_1)| < e^{x}\epsilon(\delta),$$

where $\epsilon(\delta) \to 0$ as $\delta \to 0$. Hence

$$f_1(z) = e^z + e^x\epsilon(\delta) + O\{e^{x(1-\delta)}\}. \tag{3.121}$$

By choosing δ first sufficiently small, and then x sufficiently large, we can make

$$|\epsilon(\delta) + O(e^{-\delta x})| < \tfrac{1}{2}, \tag{3.122}$$

and so

$$R(z) > \tfrac{1}{2}e^x$$

for $x \geqslant K$. In particular $f(z)$ has no zeros for these values of x. The zeros with large negative real part may be disposed of similarly.

It is plain that we have proved incidentally that $f(x)$ is bounded in any strip in which $|x|$ is bounded. This can also be proved by the second mean-value theorem.

3.21. The proof of Theorem VIII depends essentially on the following lemma.

LEMMA B. *Suppose that $F(z)$ is regular for $r \leqslant r'$, $F(0) = 1$, $k > 1$, and that $M(r')$ is the maximum of $|F(z)|$ on the circle $r = r'$. Then there is a number $H = H(k)$ such that*

$$\left|\int_{\theta_0}^{\theta} d\{\arg F(r''e^{i\theta})\}\right| < H \log M(r'), \tag{3.211}$$

for $|\theta - \theta_0| \leqslant 2\pi$ and $r'' \leqslant r'/k$.

It must be observed that if $\theta = \theta_0 + 2\pi$ the result is comparatively trivial. For we can show by Jensen's* formula that

$$n(r'') \leqslant \frac{\log M(r')}{\log (r'/r'')} \leqslant \frac{\log M(r')}{\log k};$$

and therefore

$$\int_{\theta_0}^{\theta_0+2\pi} d\{\arg F(r''e^{i\theta})\} = 2\pi n(r'') \leqslant \frac{2\pi \log M(r')}{\log k}.$$

(If a zero of $F(z)$ lies on the circle $r = \rho$, I suppose that the contour is indented so as to include the zero.)

That the case in which $|\theta - \theta_0| < 2\pi$ should be less simple is natural. Suppose that $F(z)$ is an integral function. Then if we integrate round a closed contour, the arguments of the factors belonging to the zeros outside the contour, and of all the exponential factors in the canonical product for $F(z)$, return to their initial values. When we integrate round part of a contour only, these factors have to be considered.

* See (3.221).

The proof of Lemma B is based on the proof of the following result due to Valiron.*

LEMMA C. *Suppose that the hypotheses of Lemma B are satisfied. Then there is a number $H = H(k)$ such that*

$$\log|F(r''e^{i\theta})| > -H\log M(r')$$

for some r'' such that $r'/k^2 \leqslant r'' \leqslant r'/k$.

The proofs of both lemmas require Caratheodory's† well-known inequality, Jensen's formula, and the following lemma due to Boutroux.‡

LEMMA D. *Suppose that*

$$P(z) = \left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\cdots\left(1-\frac{z}{z_n}\right),$$

where $|z_n| \leqslant r'$, and that $k > 1$. Then

$$|P(z)| > \{(k-1)k^{-4}(2e)^{-4k^3}\}^n$$

on some circles in each of the annular regions

$$r'/k \leqslant r \leqslant r', \qquad r'/k^3 \leqslant r \leqslant r'/k^2.$$

3.22. *Proof of Lemma B.* We can write

$$F(z) = P(z)e^{\Phi(z)},$$

where $P(z)$ is a polynomial formed with the zeros of $F(z)$ inside the circle $r = r'/k^{\frac{1}{2}}$, $P(0) = 1$, $\Phi(z)$ is regular inside this circle, and $\Phi(0) = 0$. It follows from Jensen's formula that

$$N(r') = \int_0^r \frac{n(u)}{u}\,du = \frac{1}{2\pi}\int_0^{2\pi} \log|F(r'e^{i\theta})|\,d\theta \leqslant \log M(r').$$

Since $n(u)$ is a non-decreasing function,

$$n\left(\frac{r'}{k^{\frac{1}{2}}}\right)\frac{\log k}{3} \leqslant \int_{r'/k^{\frac{1}{2}}}^{r'} \frac{n(u)}{u}\,du \leqslant N(r'),$$

so that

$$n\left(\frac{r'}{k^{\frac{1}{2}}}\right) \leqslant \frac{3\log M(r')}{\log k}. \tag{3.221}$$

Hence by Lemma D, there is a circle $r = r'''$ such that

$$k^{-\frac{1}{2}}r' \leqslant r''' \leqslant k^{-\frac{1}{2}}r'$$

on which

$$|P(z)| > \{M(r')\}^{-H'}.$$

* See G. Valiron, *Lectures on the General Theory of Integral Functions*, Toulouse (1923), 89. See also G. Pólya, *Journal London Math. Soc.* 1 (1926), 78 footnote.

† See G. Valiron, loc. cit., 20.

‡ P. Boutroux, *Acta Math.* 28 (1903), 97–226; see also G. Valiron, loc. cit. 80.

Hence $$|e^{\Phi(z)}| = \left|\frac{F(z)}{P(z)}\right| < \{M(r')\}^{1+H'} \tag{3.222}$$

for $r = r'''$.

Let $\mu(r)$, $\beta(r)$, and $A(r)$ be the maxima of $|\Phi(z)|$, $|\Im\{\Phi(z)\}|$, and $\Re\{\Phi(z)\}$ for $|\theta-\theta_0| \leqslant 2\pi$. By Caratheodory's inequality

$$\beta(r) \leqslant \mu(r) \leqslant \frac{4r}{k^{-\frac{1}{3}}r'-r} A(k^{-\frac{1}{3}}r'),$$

for $r < k^{-\frac{1}{3}}r'$. But it follows from (3.222) that

$$A(k^{-\frac{1}{3}}r') < A(r''') < (1+H')\log M(r'),$$

and so $$\beta(r) \leqslant (1+H')\frac{4r}{k^{-\frac{1}{3}}r'-r}\log M(r'),$$

for $r < k^{-\frac{1}{3}}r'$.

Now we can express $\arg F(z)$ in the form

$$\arg F(z) = \arg P(z) + \Im\{\Phi(z)\}.$$

It follows that

$$\int_{\theta_0}^{\theta} d\{\arg F(r''e^{i\theta})\} = \int_{\theta_0}^{\theta} d\{\arg P(r''e^{i\theta}) + \int_{\theta_0}^{\theta} d\Im\{\Phi(r''e^{i\theta})\}. \tag{3.223}$$

We have seen that the number of factors in $P(z)$ is less than $3\log M(r')/\log k$. Hence

$$\left|\int_{\theta_0}^{\theta} d\{\arg P(r''e^{i\theta})\}\right| \leqslant \frac{6\pi \log M(r')}{\log k}, \tag{3.224}$$

for $|\theta-\theta_0| \leqslant 2\pi$. Also, if $r'' \leqslant r'/k$,

$$|\Im\{\Phi(r''e^{i\theta})\}| \leqslant \beta(r'') \leqslant \beta\left(\frac{r'}{k}\right) \leqslant \frac{4r'(1+H')}{k^{\frac{1}{3}}r'-r'}\log M(r') =$$

$$= \frac{4(1+H')}{k^{\frac{1}{3}}-1}\log M(r').$$

Therefore

$$\left|\int_{\theta_0}^{\theta} d\Im\{\Phi(r''e^{i\theta})\}\right| \leqslant \frac{8\pi(1+H')}{k^{\frac{1}{3}}-1}\log M(r'). \tag{3.225}$$

Combining (3.224) and (3.225) with (3.223), we obtain (3.211).

3.23. *Proof of Theorem VIII.* We saw in 3.12 (i) that $f_1(z)$ is bounded in any strip in which $|x|$ is bounded, and (ii) that there is a K such that

$$R(z) > \tfrac{1}{2}e^x$$

for $x \geqslant K$. We can therefore find an $H_1 = H_1(K)$* such that

$$R(z) < H_1 \tag{3.231}$$

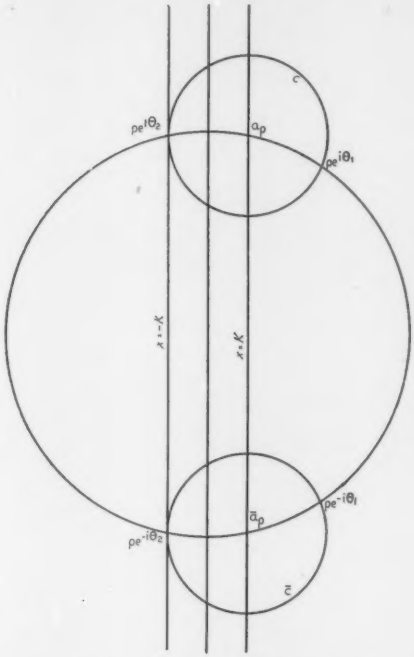
* In what follows H always denotes a number depending on K and on previous H's, and therefore ultimately on K only.

for $|x| \leqslant 10K$, and an $H_2 = H_2(K)$ such that

$$R(z) > H_2 \tag{3.232}$$

for $x = K$.

We denote the line $x = K$ by L, the circle $r = \rho$ by Γ, and their points of intersection by a_ρ and $\bar{a}_\rho$, a_ρ being above the real axis. We denote the circles $|z - a_\rho| = 2K$, $|z - \bar{a}_\rho| = 2K$ by c and $\bar{c}$ respectively.



Finally, we suppose that c cuts Γ in $\rho e^{i\theta_1}$ and $\rho e^{i\theta_2}$, and that $\bar{c}$ cuts Γ in $\rho e^{-i\theta_1}$, $\rho e^{(2\pi-\theta_2)i}$ (the first of each pair of points being that whose real part is positive). Now let

$$\frac{f_1(z)}{f_1(a_\rho)} = \frac{f_1(a_\rho + u)}{f_1(a_\rho)} = F(u), \tag{3.233}$$

and apply Lemma B to $F(u)$ with $r' = 8K$, $k = 2$, and $r'' = 2K$. It follows from the lemma, and (3.231) and (3.232), that

$$|\int d\Phi| < H_3, \tag{3.234}$$

where $H_3 = H_3(H_1, H_2) = H_3(K)$, and the integral is taken round any arc of c. There is obviously a similar inequality for the integral taken round any arc of $\bar{c}$.

Let C_1 and C_2 be the contours formed by (i) the arcs of Γ from $\rho e^{-i\theta_1}$ to $\rho e^{i\theta_1}$, and from $\rho e^{i\theta_2}$ to $\rho e^{(2\pi-\theta_2)i}$, (ii) the connecting arcs of c and $\bar{c}$, C_1 being completed by the arcs of c and $\bar{c}$ remote from the origin, and C_2 by those nearer the origin. If C is C_1 or C_2, and $n(C)$ the number of zeros of $f_1(z)$ inside or on the boundary of C, then

$$n(C) = \frac{1}{2\pi i}\int_C d\Theta = I_1 + I_2, \tag{3.235}$$

where I_1 is the part of the integral corresponding to the arcs (i), and I_2 that corresponding to the arcs (ii). It follows from (3.234) that

$$|I_2| < H_4. \tag{3.236}$$

The integral I consists of two parts, I_1' and I_1'', the first lying in the half-plane $x \geqslant K$ and the second in the half-plane $x \leqslant -K$. In I_1' we have by (3.121) and (3.122)

$$\begin{aligned} f_1(z) &= e^z + e^x\epsilon(\delta) + O\{e^{x(1-\delta)}\} \\ &= e^z(1+\zeta), \end{aligned} \tag{3.237}$$

where $|\zeta| < \frac{1}{2}$. It follows that

$$I_1' = \frac{\rho}{\pi}\sin\theta_1 + O(1) = \frac{\rho}{\pi} + O(1),$$

since θ_1 differs from $\frac{1}{2}\pi$ by $O(\rho^{-1})$. Similarly

$$I_1'' = \frac{\rho}{\pi} + O(1),$$

and so

$$I_1 = \frac{2\rho}{\pi} + O(1). \tag{3.238}$$

Combining (3.238) with (3.235) and (3.236) we see that

$$n(C) = \frac{2\rho}{\pi} + O(1).$$

Since this is true whether C is C_1 or C_2, and $n(\rho)+1$ lies between $n(C_1)$ and $n(C_2)$, we deduce the required result.

3.3. We may replace C_1 and C_2 by the lines $x = \pm K$ and the arcs of c and $\bar{c}$ joining them, since the new contours enclose the same numbers of zeros. We can then connect the lines $x = \pm K$ by the

arcs of $|z-K| = 2K$ on either side of the origin. Hence we can show that the number of zeros for which $r \leqslant \rho$, $y > 0$, is

$$\frac{\rho}{\pi} + O(1). \tag{3.31}$$

If $\phi(t)$ is real, the zeros are pairs of conjugate complex numbers, and (3.31) is a trivial deduction from Theorem VIII. Also Theorem VIII can be proved without using Lemma B. We proceed as follows:*

Replace the contours C_1 and C_2 by the rectangles $x = \pm K$, $y = \pm\rho$ and $x = \pm K$, $y = \pm(\rho - K')$, where $K' \geqslant \rho - \sqrt{\rho^2 - K^2}$. We denote the new contours by C_1' and C_2' respectively. Then we still have

$$n(C_2') \leqslant n(\rho) + 1 \leqslant n(C_1'),$$

and it is only necessary to consider $\int d\Phi$ taken along the lines

$$y = \pm\rho, \qquad y = \pm(\rho - K'),$$

since the integral taken along $x = \pm K$ can be evaluated as before.

Suppose that $\Re\{f_1(z)\}$ vanishes q times between $K + i\rho$ and $-K + i\rho$; then the interval can be divided into $q+1$ parts, throughout each of which $\Re\{f_1(z)\}$ is of constant sign. Hence in each the variation of $\Theta(z)$ is less than π, so that its total variation is less than $(q+1)\pi$. But on $y = 0$

$$\Re\{f_1(z + i\rho)\} = \tfrac{1}{2}\{f_1(z + i\rho) + f_1(z - i\rho)\} = F(z).$$

Therefore q is less than or equal to the number of zeros of $F(z)$ inside any circle enclosing the interval $-K \leqslant x \leqslant K$ of the real axis. Hence applying Jensen's formula as in (3.221) we can show that

$$|\int d\Theta| \leqslant (q+1)\pi < H_5,$$

where the integral is taken along $y = \rho$ from $K + i\rho$ to $-K + i\rho$. Similar results can obviously be obtained for $y = -\rho$, $y = \pm(\rho - K')$.

3.41. The result of Theorem VIII shows that

$$N(\rho) = \int_0^\rho \frac{n(u)}{u}\, du = \frac{2\rho}{\pi} + O(\log \rho).$$

I shall, however, prove a much more precise result.

THEOREM IX. *If $\phi(t)$ is of bounded variation, then*

$$N(\rho) = \frac{2\rho}{\pi} - \log \rho - \log|f(0)| + o(1) \tag{3.411}$$

as $\rho \to \infty$.

* Compare R. J. Backlund, *Acta Math.* 41; also J. D. Tamarkin, *Journal London Math. Soc.* 2 (1927), 66–9; and C. T. Wilder, *Trans. Amer. Math. Soc.* 18 (1917), 415–42.

This shows that 'on the average' $f(z)$ has the same number of zeros as $(\sinh z)/z$, for which

$$N(\rho) = \frac{2\rho}{\pi} - \log \rho + O\left(\frac{1}{\rho}\right).$$

The proof is based upon Jensen's formula

$$N(\rho) = \log \frac{\rho^n}{r_1 r_2 \dots r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \, d\theta - \log |f(0)|, \quad (3.412)$$

where $r_1, r_2, \dots, r_n$ are the moduli of the zeros for which $r \leqslant \rho$. It is easy to evaluate the integral in (3.412) along any arc for which $|x| \geqslant K$, but we require some additional machinery before we can deal with the strip containing the zeros.

3.42. *Proof.* By (3.121), given any $\eta > 0$, we can choose K so that

$$f_1(z) = e^z(1+\zeta), \quad (3.421)$$

where $|\zeta| < \eta$ for $x \geqslant K$. As before, let a_ρ and $\bar{a}_\rho$ be the points of intersection of $r = \rho$ and $x = K$. We have again

$$R(z) < H_6(K) = H_6(\eta), \quad (3.422)$$

for $|x| \leqslant 10K$, and

$$R(z) > H_7(K, \eta) = H_7(\eta) \quad (3.4221)$$

on $x = K$. Hence we may apply Lemmas B and C with $r' = 8K$, $k = 2$, $r'' = \gamma$ to the function $F(u)$ defined as in (3.233), and we have

$$|\int d\Theta| < H_8(\eta), \quad (3.423)$$

and

$$R(z) > -H_9(\eta), \quad (3.424)$$

on some circle $|z - a_\rho| = \gamma$, where $2K \leqslant \gamma \leqslant 4K$. There are obviously similar inequalities for some circle $|z - \bar{a}_\rho| = \gamma'$, where $2K \leqslant \gamma' \leqslant 4K$. We now denote the circles $|z - a_\rho| = \gamma$, and $|z - \bar{a}_\rho| = \gamma'$ by c and $\bar{c}$, and denote their points of intersection with Γ by $\rho e^{i\theta_1}$, $\rho e^{i\theta_2}$, $\rho e^{-i\theta_1'}$, $\rho e^{-i\theta_2'}$.

We construct C from Γ and the new c and $\bar{c}$ as in 3.2. We also construct a loop round each zero inside C, each loop consisting of a small circle of radius δ round the zero and two lines joining it to some point on one or other arc of Γ. The loops must be arranged so that they do not intersect, and so that $\rho/\pi + O(1)$* of them are joined to the arc in the right half-plane, and the rest to the arc in

* Theorem II shows that there are more than enough loops to do this, or we may consider the unsymmetrical function $e^{-z} f(z)$, and all the loops can be joined to the arc in the left half-plane.

the left half-plane. Finally, from C together with the loops we construct a closed contour $D(\delta)$ inside which $f(z)$ has no zeros. It follows that $\log f(z)$ is regular inside $D(\delta)$, and we can apply Cauchy's theorem.

We make $\delta \to 0$, and take the real part. We write

$$N(C) = \log \frac{\rho^{n'}}{r_1 r_2 \dots r_{n'}},$$

where $r_1, r_2, \dots r_{n'}$ are the moduli of the zeros inside or on the boundary of C, and we have

$$\begin{aligned} N(C) &= \Re\left\{\frac{1}{2\pi i}\int_C \frac{\log f(z)}{z}\,dz\right\} - \log|f(0)| \\ &= \Re\left\{\frac{1}{2\pi i}\int_C \frac{\log f_1(z)}{z}\,dz\right\} - \Re\left\{\frac{1}{2\pi i}\int_C \frac{\log z}{z}\,dz\right\} - \log|f(0)| \\ &= J - \log\rho - \log|f(0)|. \end{aligned} \tag{3.425}$$

Let
$$J = I_1 + I_2 + I_3 + I_4, \tag{3.4251}$$

where I_1 and I_3 denote the parts of the integral corresponding to the arcs of $\bar{c}$ and c respectively, and I_2 and I_4 the parts corresponding to the arcs of Γ in the right and left half-planes respectively. Then by (3.421)

$$I_2 = \frac{1}{2\pi}\int_{-\theta_1'}^{\theta_1} \log R(\rho e^{i\theta})\,d\theta = \frac{\rho}{2\pi}\int_{-\theta_1'}^{\theta_1} \cos\theta\,d\theta + \epsilon(\eta) = \frac{\rho}{\pi} + \epsilon(\eta), \tag{3.426}$$

where $\epsilon(\eta) \to 0$ as $\eta \to 0$. Similarly,

$$I_4 = \frac{\rho}{\pi} + \epsilon(\eta). \tag{3.427}$$

Also
$$|I_1| < \frac{1}{2\pi}\int_{\bar{c}} \frac{|\log R(z)| + |\Theta(z)|}{|z|}\,|dz|. \tag{3.428}$$

It follows from (3.424) and (3.422) that $\log R(z)$ is bounded on c, and similarly on $\bar{c}$. It follows from (3.423) that $\Theta(z)$ is bounded on the arcs of c and $\bar{c}$ considered, provided that it is bounded at the beginning of each arc, i.e. at $\rho e^{i\theta_1}$ and $\rho e^{-i\theta_2'}$. We may suppose that $|\Theta(z)| \leqslant \pi$ at $\rho e^{-i\theta_2}$. Then by (3.428)

$$|I_1| \leqslant H_{10}\int_{\bar{c}} \left|\frac{dz}{z}\right| < \frac{H_{11}}{\rho}. \tag{3.429}$$

A similar inequality holds for $|I_3|$, provided that $\Theta(z)$ remains bounded at $\rho e^{i\theta_1}$ after z has described the part of the contour $D(\delta)$

from $\rho e^{-i\theta_2'}$ to $\rho e^{i\theta_1}$. We have seen that $\Theta(z)$ is bounded on $\bar{c}$ and therefore at $\rho e^{-i\theta_1'}$. As z moves along the arc of Γ from $\rho e^{-i\theta_1'}$ to $\rho e^{i\theta_1}$, $\Theta(z)$ increases by $2\rho+O(1)$, but in describing each loop it is reduced by 2π. It follows from the arrangement of the loops that $\Theta(z)$ is bounded at $\rho e^{i\theta_1}$, and hence

$$|I_3| < H_{12}/\rho. \tag{3.4291}$$

Now combining (3.426), (3.427), (3.429), and (3.4291) with (3.425), we have

$$N(C) = \frac{2\rho}{\pi} - \log\rho - \log|f(0)| + \epsilon(\eta) + \frac{H_{13}}{\rho},$$

where H_{13} depends on η, and is bounded as $\rho \to \infty$. Hence

$$N(C) = \frac{2\rho}{\pi} - \log\rho - \log|f(0)| + o(1).$$

It follows from (3.221), and (3.422) and (3.4221) that the number of zeros inside c and $\bar{c}$ is bounded as $\rho \to \infty$, and so

$$|N(C) - N(\rho)| < H_{14} \operatorname{Max} |\log(\rho/r_p)|,$$

where r_p runs through the zeros inside c and $\bar{c}$. But $|r_p - \rho| < 3K$, so that

$$|N(C) - N(\rho)| = O(1/\rho).$$

The required result follows at once.

It may be observed that the result could be much improved if the approximation in (3.421) as $x \to \infty$, and therefore the error in (3.426) and (3.427), were better. For the error terms arising from c and $\bar{c}$ are $O(1/\rho)$.

$\phi(t)$ continuous.

4.1. The methods used for Theorems VIII and IX can also be applied when $\phi(t)$ satisfies a Lipschitz condition, or, more generally, when $\phi(t)$ is any continuous function. We define the 'modulus of continuity' as follows:*

Let $\phi(t)$ be continuous in the interval (a, b); consider any two points t_1, t_2 in (a, b), and let $\omega(\delta)$ be the maximum of $|\phi(t_2) - \phi(t_1)|$ for all points t_1, t_2 such that $|t_2 - t_1| < \delta$. Then $\omega(\delta)$ is the *modulus of continuity* of $\phi(t)$ in (a, b).

$\omega(\delta)$ is a continuous, non-decreasing function of δ and tends to

* See Ch.-V. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, Gauthier-Villars (1919), 7.

zero with δ. If $\omega(\delta) = O(\delta^\alpha)$, where $0 < \alpha \leqslant 1$, $\phi(t)$ satisfies a Lipschitz condition of order α. If $\omega(\delta) = o(\delta)$, $\phi(t)$ reduces to a constant.

The results for $\alpha = 1$ are included in those of Theorems VII and VIII, but not in that of IX. The natural generalization of Theorem VII is

THEOREM X. *If $\phi(t)$ is continuous in $(-1, 1)$, then all the zeros of $f(z)$ lie inside the region $|x| < K r\omega(1/r)$ for some K.*

The $[r]$th Fourier coefficient of $\phi(t)$ is of the form $O\{\omega(1/r)\}$, and we have, as we should expect,

THEOREM XI. *If $\phi(t)$ is continuous in $(-1, 1)$, then*

$$n(\rho) = \frac{2\rho}{\pi} + O\left\{\rho\omega\left(\frac{1}{\rho}\right)\right\},$$

as $\rho \to \infty$.

4.2. *Proof of Theorem X.* Divide* the interval $(-1, 1)$ into $2m$ equal parts, and let

$$\phi(t) = \phi\left(\frac{p}{m}\right) - \psi_p(t) \qquad \left(\frac{p-1}{m} \leqslant t \leqslant \frac{p}{m}\right).$$

Then
$$|\psi_p(t)| \leqslant \omega\left(\frac{1}{m}\right).$$

Suppose that $x > 0$; then

$$\begin{aligned} f(z) &= \sum_{p=-m+1}^{m} \int_{(p-1)/m}^{p/m} e^{zt}\left\{\phi\left(\frac{p}{m}\right) - \psi_p(t)\right\} dt \\ &= \sum \phi\left(\frac{p}{m}\right) \int_{(p-1)/m}^{p/m} e^{zt}\, dt - \sum \int_{(p-1)/m}^{p/m} e^{zt}\psi_p(t)\, dt \\ &= I_1 + I_2. \end{aligned} \tag{4.21}$$

Now
$$\begin{aligned} |I_2| &\leqslant \sum_{p=-m+1}^{m} \int_{(p-1)/m}^{p/m} e^{xt}\omega\left(\frac{1}{m}\right) dt \\ &= \omega\left(\frac{1}{m}\right) \int_{-1}^{1} e^{xt}\, dt = \omega\left(\frac{1}{m}\right) \frac{e^x - e^{-x}}{x}. \end{aligned} \tag{4.22}$$

* I am indebted to Professor Titchmarsh for suggesting that I should subdivide the interval instead of attempting to use fractional orders of integration, as I had done previously without success.

Also
$$I_1 = \sum_{p=0}^{2m-1} \frac{\phi(1-p/m)}{z} \{e^{z(1-p/m)} - e^{z(1-(p-1)/m)}\}$$
$$= \frac{e^z}{z} - \frac{e^z}{z} \sum_{1}^{2m-1} e^{-pz/m} \left\{\phi\left(1-\frac{p-1}{m}\right) - \phi\left(1-\frac{p}{m}\right)\right\} - \\ - \phi\left(-1+\frac{1}{m}\right)\frac{e^{-z}}{z}$$
$$= \frac{e^z}{z} - \phi\left(-1+\frac{1}{m}\right)\frac{e^{-z}}{z} + \frac{e^z}{z} I_3, \tag{4.23}$$

where
$$|I_3| < \sum_{p=1}^{\infty} e^{-px/m} \omega\left(\frac{1}{m}\right) = \omega\left(\frac{1}{m}\right) \frac{e^{-x/m}}{1-e^{-x/m}} < H_{15} \omega\left(\frac{1}{m}\right) \frac{m}{x}, \tag{4.24}$$

provided that $m > m_0(x)$. Hence, combining (4.22), (4.23), and (4.24) with (4.21), and putting $m = [r]$, we have
$$f_1(z) = e^z - \phi\left(-1+\frac{1}{m}\right) e^{-z} + e^z O\left\{\frac{r\omega(1/r)}{x}\right\}. \tag{4.25}$$

By choosing K sufficiently large we can make
$$\left|\phi\left(-1+\frac{1}{m}\right) e^{-2z} + O\left(\frac{r\omega(1/r)}{x}\right)\right| < \tfrac{1}{2},$$

and so
$$R(z) > \tfrac{1}{2} e^x, \tag{4.26}$$

for $x \geqslant Kr\omega(1/r)$. In particular $f(z)$ has no zeros satisfying this condition, and the zeros with large negative real part may be disposed of similarly.

4.3. *Proof of Theorem XI.* In 4.2, if $x \geqslant 0$, we have
$$|I_3| \leqslant \omega\left(\frac{1}{m}\right) \sum_{p=1}^{2m-1} e^{-px/m} = O\left\{r\omega\left(\frac{1}{r}\right)\right\},$$

and
$$|I_2| \leqslant \omega\left(\frac{1}{m}\right) \int_{-1}^{1} e^{xt}\, dt = O\left\{e^x \omega\left(\frac{1}{r}\right)\right\}.$$

Hence
$$R(z) = O\{e^{|x|} r\omega(1/r)\}. \tag{4.31}$$

The inequality obviously holds also for $x \leqslant 0$. Hence we can find H_{16} such that
$$R(z) < e^{H_{16} r\omega(1/r)} \tag{4.32}$$

for $|x| \leqslant 10Kr\omega(1/r)$. It follows from (4.26) that

$$R(z) > \tfrac{1}{2}e^{Kr\omega(1/r)} \tag{4.33}$$

for $x = Kr\omega(1/r)$.

We denote the line $x = Kr\omega(1/r)$ by L, the circle $r = \rho$ by Γ, and their points of intersection by a_ρ and $\bar{a}_\rho$. We denote the circles $|z-a_\rho| = 2K\rho\omega(1/\rho)$, and $|z-\bar{a}_\rho| = 2K\rho\omega(1/\rho)$ by c and $\bar{c}$, and we define $F(u)$ as in (3.233). Then by (4.33) and (4.32)

$$|F(u)| < e^{H_{17}\rho\omega(1/\rho)} \tag{4.34}$$

for $|z-a_\rho| \leqslant 8K\rho\omega(1/\rho)$. Hence we can apply Lemma B with $r' = 8K\rho\omega(1/\rho)$, $k = 2$, $r'' = 2K\rho\omega(1/\rho)$, and we have

$$\left|\int d\Theta\right| = \left|\int d\{\arg F(u)\}\right| < H_{18}\rho\omega\left(\frac{1}{\rho}\right), \tag{4.35}$$

where the integral is taken round any arc of c. A similar inequality holds for $\bar{c}$.

Let C_1 and C_2 be the contours formed from c, $\bar{c}$, and Γ as in Theorem VIII, and let C denote either C_1 or C_2. As before (3.235) holds, where I_1 is the part of the integral corresponding to the arcs of Γ, and I_2 that corresponding to the arcs of c and $\bar{c}$. It follows from (4.35) that

$$|I_2| < H_{19}\rho\omega(1/\rho). \tag{4.36}$$

I_1 consists of two parts, I_1' and I_1''. In I_1' $x > Kr\omega(1/r)$, and in I_1'' $x < -Kr\omega(1/r)$. In I_1'' therefore, by (4.25),

$$f_1(z) = e^z\left\{1+O\left(\frac{r\omega(1/r)}{x}\right)+O(e^{-2x})\right\} = e^z(1+\zeta),$$

where $|\zeta| < \frac{1}{2}$. It follows that

$$I_1' = \frac{\rho}{\pi}\sin\theta_1 + O(1) = \frac{\rho}{\pi}+O(1)+O\left\{\rho\omega^2\left(\frac{1}{\rho}\right)\right\}, \tag{4.37}$$

for θ_1 differs from $\frac{1}{2}\pi$ by $O\{\omega(1/\rho)\}$. Similarly

$$I_1'' = \frac{\rho}{\pi}+O(1)+O\left\{\rho\omega^2\left(\frac{1}{\rho}\right)\right\}. \tag{4.38}$$

Combining (4.36), (4.37), and (4.38) with (3.235) we have

$$n(C) = \frac{2\rho}{\pi}+O\left\{\rho\omega\left(\frac{1}{\rho}\right)\right\},$$

and the required result follows as before.

4.4. As in 3.3 we can replace C_1 and C_2 by contours formed from

the lines $x = \pm Kr\omega(1/r)$ and the connecting arcs of c and $\bar{c}$. We can also show that the number of zeros for which $r \leqslant \rho$, $y > 0$ is

$$\frac{\rho}{\pi} + O\left\{\rho\omega\left(\frac{1}{\rho}\right)\right\}.$$

4.5. In place of Theorem IX we have

THEOREM XII. *If $\phi(t)$ is continuous in the interval $(-1, 1)$, then*

$$N(\rho) = \frac{2\rho}{\pi} - \log\rho - \log|f(0)| + O\left\{\rho\omega^2\left(\frac{1}{\rho}\right)\right\} + O\left\{\omega\left(\frac{1}{\rho}\right)\log\frac{1}{\omega(1/\rho)}\right\}$$

as $\rho \to \infty$.

Proof. Since the function $F(u)$ of 4.3 satisfies (4.34) we can apply Lemmas B and C with $r' = 8K\rho\omega(1/\rho)$, $k = 2$, $r'' = \gamma$. We thus obtain

$$|\int d\Theta| < H_{20}(K)\rho\omega\left(\frac{1}{\rho}\right), \tag{4.51}$$

and

$$\log R(z) > -H_{21}(K)\rho\omega\left(\frac{1}{\rho}\right), \tag{4.52}$$

on some circle $|z - a_\rho| = \gamma$, where

$$2K\rho\omega(1/\rho) < \gamma < 4K\rho\omega(1/\rho). \tag{4.53}$$

Similar inequalities hold for some circle $|z - \bar{a}_\rho| = \gamma'$, where γ' also satisfies (4.53). We denote these circles by c and $\bar{c}$, and construct C and $D(\delta)$ as in 3.42.

As before, we obtain (3.425) and (3.4251), and by (4.25) we can put

$$f_1(z) = e^z\left\{1 + O\left(\frac{r\omega(1/r)}{x}\right)\right\} = e^z\left\{1 + O\left(\frac{\omega(1/r)}{\cos\theta}\right)\right\}$$

in I_2. It follows that

$$\begin{aligned} I_2 &= \frac{\rho}{2\pi}\int_{-\theta_1'}^{\theta_1} \cos\theta\, d\theta + O\left\{\omega\left(\frac{1}{\rho}\right)\int_{-\theta_1'}^{\theta_1} \sec\theta\, d\theta\right\} \\ &= \frac{\rho}{\pi} + O\left\{\rho\omega^2\left(\frac{1}{\rho}\right)\right\} + O\left\{\omega\left(\frac{1}{\rho}\right)\log\frac{1}{\omega(1/\rho)}\right\}, \end{aligned} \tag{4.54}$$

since θ_1 differs from $\frac{1}{2}\pi$ by $O\{\omega(1/\rho)\}$. Similarly,

$$I_4 = \frac{\rho}{\pi} + O\left\{\rho\omega^2\left(\frac{1}{\rho}\right)\right\} + O\left\{\omega\left(\frac{1}{\rho}\right)\log\frac{1}{\omega(1/\rho)}\right\}. \tag{4.55}$$

It follows from (4.32) and (4.52) that

$$|\log R(z)| < H_{22}\rho\omega(1/\rho)$$

on c and $\bar{c}$, and it follows from (4.51) that

$$|\Theta(z)| < H_{23}\rho\omega(1/\rho)$$

on the arcs of c and $\bar{c}$ considered, provided that the inequality is satisfied at the beginning of each arc. This is easily seen to be the case by supposing that $|\Theta(z)| \leqslant \pi$ at $\rho e^{-i\theta_2'}$, and considering its variation along $D(\delta)$. Hence, using (3.428), we have

$$|I| < H_{24}\rho\omega\left(\frac{1}{\rho}\right)\int_{\bar{c}}\left|\frac{dz}{z}\right| < H_{25}\rho\omega^2\left(\frac{1}{\rho}\right), \tag{4.56}$$

and a similar inequality for $|I_3|$.

Combining (4.54), (4.55), and (4.56) with (3.425) and (3.4251), we have

$$N(C) = \frac{2\rho}{\pi} - \log\rho - \log|f(0)| + O\left\{\rho\omega^2\left(\frac{1}{\rho}\right)\right\} + O\left\{\omega\left(\frac{1}{\rho}\right)\log\frac{1}{\omega(1/\rho)}\right\}.$$

By (3.221) and (4.34) the number of zeros inside c and $\bar{c}$ is less than $H_{26}\rho\omega(1/\rho)$, so that

$$|N(C) - N(\rho)| < H_{26}\rho\omega\left(\frac{1}{\rho}\right)\mathrm{Max}\left|\log\frac{\rho}{r_p}\right|,$$

where r_p runs through the moduli of the zeros inside c and $\bar{c}$. But $|r_p - \rho| < 3K\rho\omega(1/\rho)$, and therefore

$$|N(C) - N(\rho)| = O\{\rho\omega^2(1/\rho)\}.$$

Hence we have the result of Theorem XII.

4.6. If $\omega(\delta) > K'\delta\log\delta$, the second error term may be neglected. In this case the result of Theorem XI may be deduced from that of Theorem XII. For suppose that $k > 1$ and nearly equal to 1. Then

$$n(\rho)\log k \leqslant \int_{\rho}^{k\rho}\frac{n(u)}{u}\,du \leqslant N(k\rho) - N(\rho) =$$

$$= \frac{2\rho}{\pi}(k-1) - \log k + O\left\{\rho\omega^2\left(\frac{1}{\rho}\right)\right\},$$

and hence

$$n(\rho) \leqslant \frac{2\rho(k-1)}{\pi\log k} - 1 + O\left\{\frac{\rho\omega^2(1/\rho)}{\log k}\right\}.$$

Put $k = 1 + \omega(1/\rho)$; then

$$\frac{k-1}{\log k} = \frac{1}{1 + O\{\omega(1/\rho)\}} = 1 + O\left\{\omega\left(\frac{1}{\rho}\right)\right\}.$$

It follows that

$$n(\rho) \leqslant \frac{2\rho}{\pi} + O\left\{\rho\omega\left(\frac{1}{\rho}\right)\right\}.$$

The opposite inequality is obtained by considering

$$\int_{\rho/k}^{\rho}\frac{n(u)}{u}\,du.$$

Exceptions to the rule

5. The analogy between the error term for $n(\rho)$ and the order of the Fourier coefficients does not by any means always hold. This is because the behaviour of $\phi(t)$ in the middle of the interval $(-1, 1)$ has little effect on $|f(z)|$ when $|x|$ is large. In fact we have

THEOREM XIII. *If $\phi(t)$ is continuous in $(-1, -1+\eta)$ and $(1-\eta, 1)$, then the results of Theorems X, XI, and XII hold, provided that we replace $\omega(1/\rho)$ in each result by*

$$\omega_1\left(\frac{1}{\rho}\right) = \operatorname{Max}\left\{\omega\left(\frac{1}{\rho}\right); \frac{\log\rho}{\rho}\right\}.$$

For $x > 0$ we write

$$f(z) = \int_{-1}^{1-\eta} e^{zt}\phi(t)\,dt + \int_{1-\eta}^{1} e^{zt}\phi(t)\,dt = I_1 + I_2. \tag{5.1}$$

As in 4.2

$$I_2 = \frac{e^z}{z}\left\{1 + O\left(\frac{r\omega(1/r)}{x}\right)\right\}. \tag{5.2}$$

It follows from Lemma A that

$$I_1 = o\{e^{(1-\eta)x}\} = o\left\{\frac{e^x}{r}.re^{-\eta x}\right\} = o\left\{\frac{e^x}{r}\right\},$$

for $x = (\log r)/\eta$, and *a fortiori*

$$I_1 = O\left\{\frac{e^x \log r}{rx}\right\}, \tag{5.3}$$

for $x \geqslant (\log r)/\eta$. Combining (5.2) and (5.3) with (5.1), we have

$$f_1(z) = e^z\left\{1 + O\left(\frac{r\omega_1(1/r)}{x}\right)\right\}.$$

The result of Theorem X follows as before with ω_1 instead of ω.

In order to obtain the results of Theorems XI and XII we have in addition to use (4.32), i.e. we have to show that

$$R(z) < e^{H_{16}r\omega_1(1/r)} \tag{5.4}$$

for $|x| \leqslant 10Kr\omega_1(1/r)$. We need only consider the integral from $-1+\eta$ to $1-\eta$. For the usual arguments show that

$$|zI_2| < e^{H_{27}r\omega_1(1/r)}, \tag{5.5}$$

and a similar inequality holds for the integral from -1 to $-1+\eta$.

For the values of x concerned we have

$$\begin{aligned} z\int_{-1+\eta}^{1-\eta} e^{zt}\phi(t)\,dt &= o\{re^{|x|(1-\eta)}\} \\ &= o\{e^{10K(1-\eta)r\omega_1(1/r)+\log r}\} = o\{e^{H_{28}r\omega_1(1/r)}\}. \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6), we have (5.4). The rest of the proofs of Theorems XI and XII can now be applied without further difficulty.

It remains to give an example of a function for which the analogy fails. Consider

$$\psi(t)=\sum_{m=1}^{\infty}\frac{e^{2^m it}}{m^2}.$$

The series is uniformly convergent for $|t|\leqslant\pi$, and therefore it is the Fourier series of its sum. But, when n is of the form 2^m,

$$a_n=\frac{1}{m^2}=\left(\frac{\log 2}{\log n}\right)^2.$$

Now let

$$\begin{aligned}\Phi(t)&=1 && (-1<t<-1+\eta)\\ &=\psi\left(\frac{\pi}{1-\eta}t\right) && (-1+\eta<t<1-\eta)\\ &=1 && (1-\eta<t<1).\end{aligned}$$

Then by Theorem XIII

$$n(\rho)=\frac{2\rho}{\pi}+O(\log\rho),$$

but, when $[\rho]=2^m$,

$$a_{[\rho]}\sim\left(\frac{\log 2}{\log[\rho]}\right)^2,$$

and so the analogy fails.

A GENERALIZATION OF THE QUADRATIC DIFFERENTIAL FORM

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WITH each point $x = (x^1\ x^2\ \dots\ x^n)$ of a space of n dimensions there is associated a flat space of n dimensions whose co-ordinates are the sets of differentials $dx = (dx^1\ dx^2\ \dots\ dx^n)$. We shall refer to these spaces in which the co-ordinates are sets of differentials as *tangent spaces*. They are affine spaces because their co-ordinates undergo an affine transformation,

$$d\bar{x}^i = v^i_j dx^j, \qquad \text{where} \quad v^i_j = \frac{\partial \bar{x}^i}{\partial x^j},$$

whenever the co-ordinates x of the 'underlying space' are subjected to an arbitrary analytic transformation,

$$\bar{x}^i = \bar{x}^i(x).$$

The group in the tangent space is, indeed, the subgroup of the affine group which leaves the origin $dx = 0$ invariant. This point, $dx = 0$, will be called the *point of contact* of the affine space with the underlying space.

A quadratic differential form determines in each tangent space a quadratic cone,

$$a_{ij} dx^i dx^j = 0,$$

with its vertex at the origin. The generalization which we propose amounts to replacing these cones by non-degenerate quadrics,

$$a_{ij} dx^i dx^j + a_i dx^i + a_0 = 0.$$

The theory of these quadrics is a chapter in the generalized projective geometry.* On account of the relationship between this theory in n dimensions and the generalized affine geometry of $n+1$ dimensions, the formalism runs closely parallel with that of the quadratic differential form and many results can be taken over directly. Moreover, the formulas in four dimensions are in part identical with those of the Five-dimensional Relativity of T. Kaluza,

* For an introduction to this subject and references to literature by Weyl, Cartan, Schouten, Eisenhart, Thomas, and others, see a paper entitled 'Generalized Projective Geometry' in the *Journal of the London Mathematical Society*, vol. 4 (1929), pp. 140–60. This paper will be referred to below as G.P.

O. Klein, and others. This relationship will be dealt with in a paper which is being prepared in collaboration with Mr. B. Hoffmann, but the present paper does not presuppose any knowledge of it, and such formulas as we require are either derived in the text or taken from well-known general theory.

1. **Homogeneous co-ordinates in the tangent spaces.** In each tangent affine space points at infinity can be introduced in the usual manner so that it can be regarded as immersed in a tangent projective space. The points of the latter will be designated by means of homogeneous co-ordinates having a factor of proportionality which we shall write in the form $\exp(-x^0)$. The parameter x^0 (which is really the negative of the logarithm of the factor of proportionality) will be referred to as 'the factor'—following a convenient terminology introduced by Mr. Whitehead.*

The invariant theory of homogeneous co-ordinates must take account of a change of factor,

$$\bar{x}^0 = x^0 + \log \rho(x^1\ x^2\ \dots\ x^n), \tag{1.1}$$

as well as of a change of co-ordinates,

$$\bar{x}^i = \bar{x}^i(x^1\ x^2\ \dots\ x^n). \tag{1.2}$$

The projective tensors defined in G.P. are a particularly simple class of invariants with respect to these transformations. One of the simplest projective tensors is a projective scalar of index 1†. This is an invariant which has one component,

$$A = e^{x^0} f(x^1\ x^2\ \dots\ x^n),$$

for each co-ordinate system and factor,‡ and whose law of transformation is

$$\bar{A} = A.$$

Thus in each co-ordinate system a projective scalar has an infinity of components, for a change of factor (1.1) changes A into

$$\bar{A} = e^{\bar{x}^0} \frac{f(x)}{\rho(x)}.$$

* J. H. C. Whitehead, 'On a class of projectively flat affine spaces'. This paper will be referred to below as P.F.

† I am using the word 'index' instead of 'weight N' which was used in G.P., so as to reserve the word weight for the 'weight M' of that paper, i.e. the weight of a relative tensor.

‡ A co-ordinate system and factor together constitute a representation $(x^0)+(x)$ according to Mr. Whitehead, P.F. § 3.

In one representation it is clear that we will have

$$\bar{A} = e^{\bar{x}^0}.$$

Let us make the convention that Greek indices run over the range $0, 1, 2, \ldots, n$, whereas Latin ones take on the values $1, 2, \ldots, n$ only. Let $\bar{A}$ be the component of the projective scalar A in the representation $(\bar{x}^0)+(\bar{x})$, obtained by a change of co-ordinate system as well as of factor from the representation $(x^0)+(x)$. Then

$$\frac{\partial \bar{A}}{\partial \bar{x}^\alpha} = \frac{\partial A}{\partial x^\beta}\frac{\partial x^\beta}{\partial \bar{x}^\alpha}.$$

By reference to G.P. it is evident that $\partial A/\partial x^\alpha$ are the components of a projective vector of index 1 and

$$A_\alpha = \frac{\partial \log A}{\partial x^\alpha} \tag{1.3}$$

are the components of a projective vector of index 0. Since $\partial A/\partial x^0 = A$, we have $A_0 = 1$.

We now introduce homogeneous co-ordinates in the tangent spaces by the formulas,

$$X^i = \frac{dx^i}{A}, \tag{1.4}$$

$$X^0 = \frac{1 - A_i dx^i}{A}. \tag{1.5}$$

The co-ordinates X^α so defined satisfy the condition

$$A_\alpha X^\alpha = \frac{1}{A}. \tag{1.6}$$

The tangent affine space becomes a projective one, if we admit ideal points whose co-ordinates satisfy the equation of the hyperplane

$$A_\alpha X^\alpha = 0. \tag{1.7}$$

With this convention every set of $n+1$ numbers $(X^0\, X^1 \ldots X^n)$ is a set of homogeneous co-ordinates of some point of a tangent projective space. Moreover, the homogeneous co-ordinates are transformed by the rule,

$$\bar{X}^\alpha = X^\beta \frac{\partial \bar{x}^\alpha}{\partial x^\beta},$$

under an arbitrary change of factor and co-ordinate system. Hence they are components of projective vectors. Since A appears in the denominators in (1.4) and (1.5), these projective vectors are of index -1.

This introduction of homogeneous co-ordinates is made in such a way that the coefficients of the equation of the hyperplane at infinity in each tangent space shall be the components of the projective gradient of A. We could if we wished replace A_α by an arbitrary projective vector B_α, so that the equation of the hyperplanes at infinity would be

$$B_\alpha X^\alpha = 0.$$

But in this case there would not exist a representation in which the equations of the hyperplanes at infinity of all tangent spaces are

$$X^0 = 0$$

simultaneously. When A_α is a gradient this happens in any representation for which

$$A = e^{x^0}.$$

2. **Equation of a quadric in the tangent space.** The components of any covariant projective vector B_α are the coefficients of the equation of a hyperplane in each tangent space,

$$B_\alpha X^\alpha = 0.$$

This equation represents the same hyperplane no matter what representation of the vector B_α is used, for the left-hand member is unaltered by any change of representation. In non-homogeneous co-ordinates the equation is

$$(B_0 A_i - B_i)dx^i = B_0.$$

In like manner the components of any covariant projective tensor $G_{\alpha\beta}$ are the coefficients of a quadric,

$$G_{\alpha\beta} X^\alpha X^\beta = 0,$$

in each tangent space. Since the left-hand member is unaltered by changes of representation, any point X which satisfies this equation in one representation does so in all representations. Hence there is a unique quadric determined by the projective tensor $G_{\alpha\beta}$—always with respect to a definite choice of the scalar A which determines the relation between homogeneous and non-homogeneous co-ordinates. In non-homogeneous co-ordinates the equation of the quadric is

$$(G_{ij} - A_i G_{0j} - A_j G_{0i} + A_i A_j G_{00})dx^i dx^j + 2(G_{0i} - G_{00} A_i)dx^i + G_{00} = 0.$$

3. **Projective tensor of second degree.** Let us denote the index of an arbitrary covariant projective tensor $G_{\alpha\beta}$ of degree 2 by $2N$. From the law of transformation of such a projective tensor we see that G_{00} is a projective scalar of index $2N$. Let us write

$$\Phi^2 = G_{00},$$

so that Φ represents a projective scalar of index N. Then

$$\gamma_{\alpha\beta} = \frac{G_{\alpha\beta}}{G_{00}}$$

is a projective tensor of index zero such that

$$\gamma_{00} = 1.$$

Again, from the law of transformation, if we define

$$\phi_\alpha = \gamma_{\alpha 0},$$

ϕ_α is a covariant projective vector of weight zero. Hence

$$g_{\alpha\beta} = \gamma_{\alpha\beta} - \phi_\alpha \phi_\beta$$

is a projective tensor for which

$$g_{\alpha 0} = g_{0\alpha} = 0.$$

Again, from the law of transformation it follows that the quantities g_{ij} are the components of an affine tensor. Thus the projective tensor $G_{\alpha\beta}$ determines and is determined by a projective scalar Φ, a projective vector ϕ_α, and an affine tensor g_{ij}, according to the formula,

$$G_{\alpha\beta} = \Phi^2(g_{\alpha\beta} + \phi_\alpha \phi_\beta). \tag{3.1}$$

The geometrical meaning of this equation is easily seen. The point of contact of the tangent space with the underlying space has homogeneous co-ordinates

$$X^\alpha = e^{-x^0}\delta^\alpha_0.$$

Hence an equation of the polar hyperplane of the point of contact is

$$\phi_\alpha X^\alpha = 0,$$

and an equation of the tangent cone with this point as vertex is

$$g_{\alpha\beta} X^\alpha X^\beta = 0.$$

In non-homogeneous co-ordinates these equations become

$$(\phi_i - A_i)dx^i + 1 = 0$$

and

$$g_{ij}dx^i dx^j = 0$$

respectively.

Referring to the equation at the end of the last section, we see that the equation of the quadric in non-homogeneous co-ordinates is

$$(g_{ij} + \psi_i \psi_j)dx^i dx^j + 2\psi_i dx^i + 1 = 0 \tag{3.2}$$

or

$$g_{ij}dx^i dx^j + (\psi_i dx^i + 1)^2 = 0,$$

where ψ_i is an affine vector defined by the equations

$$\psi_i = \phi_i - A_i. \tag{3.3}$$

Since $\phi_0 = A_0 = 1$,

$$\psi_\alpha = \phi_\alpha - A_\alpha$$

defines a projective vector whose null component is zero.

Let us denote the $(n+1)$-rowed determinant of the quantities $\gamma_{\alpha\beta}$ by γ and the n-rowed determinant of the quantities g_{ij} by g. Then a simple computation shows that

$$\gamma = g. \tag{3.4}$$

We also consider the contravariant affine tensor g^{ij} defined by the equations

$$g^{ij}g_{ik} = \delta^j_k$$

as well as the contravariant projective tensors $\gamma^{\alpha\beta}$ and $G^{\alpha\beta}$ defined by

$$\gamma^{\alpha\sigma}\gamma_{\alpha\tau} = \delta^\sigma_\tau \text{ and } G^{\alpha\sigma}G_{\alpha\tau} = \delta^\sigma_\tau. \tag{3.4}$$

Obviously

$$G^{\alpha\beta} = \frac{1}{\Phi^2}\gamma^{\alpha\beta}, \tag{3.5}$$

and $\gamma^{\alpha\beta}$ and $G^{\alpha\beta}$ are of indices 0 and $-2N$ respectively.

For all affine tensors we shall raise and lower indices in the usual manner by means of g^{ij} and g_{ij}, and for projective tensors by means of $\gamma^{\alpha\beta}$ and $\gamma_{\alpha\beta}$. Thus, for example,

$$\gamma_{\alpha\beta}\delta^\beta_0 = \phi_\alpha, \text{ and hence } \phi^\alpha = \delta^\alpha_0. \tag{3.6}$$

It is a corollary of these equations that

$$\gamma^{\alpha\beta}\phi_\alpha\phi_\beta = 1.$$

If we make the definition

$$\phi_{\sigma\tau} = \frac{1}{2}\left(\frac{\partial\phi_\sigma}{\partial x^\tau} - \frac{\partial\phi_\tau}{\partial x^\sigma}\right), \tag{3.7}$$

it follows that

$$\phi_{\sigma 0} = \phi_{0\sigma} = 0,$$

and it is evident that ϕ_{ij} is an affine tensor. Hence we raise and lower its indices by means of the g's. Thus

$$\phi^i_j = \tfrac{1}{2}g^{ik}\left(\frac{\partial\phi_k}{\partial x^j} - \frac{\partial\phi_j}{\partial x^k}\right),$$

and so

$$\phi^i_i = 0.$$

We adopt the convention that, where the contrary is not stated, it is always the *first* index which is raised or lowered. This determines the sign of ϕ^i_j. In general, raising and lowering Greek indices amounts geometrically to polarizing with respect to the quadric

$$\gamma_{\alpha\beta}X^\alpha X^\beta = 0,$$

and raising and lowering of Latin indices amounts to polarizing with respect to the degenerate quadric

$$g_{ij}dx^i dx^j = 0.$$

The relation between $\gamma^{\alpha\beta}$ and the affine tensor may be found as follows: By (3.4) and (3.6),

$$\delta^j_k = \gamma^{\alpha j}\gamma_{\alpha k} = \gamma^{\alpha j}(g_{\alpha k} + \phi_\alpha \phi_k) = \gamma^{ij} g_{jk}.$$

Hence we have
$$\gamma^{ij} = g^{ij}. \tag{3.8}$$

Again,
$$0 = \gamma^{\alpha i}\gamma_{\alpha 0} = g^{ji}\phi_j + \gamma^{0i}.$$

It follows that
$$\gamma^{i0} = \gamma^{0i} = -g^{ji}\phi_j. \tag{3.9}$$

We do not write $g^{ji}\phi_j = \phi^i$ because the quantities ϕ^i are not the components of an affine tensor. Finally

$$1 = \gamma^{\alpha 0}\gamma_{\alpha 0} = -g^{ji}\phi_j \phi_i + \gamma^{00}.$$

Hence
$$\gamma^{00} = 1 + g^{ij}\phi_i \phi_j. \tag{3.10}$$

The affine tensor g_{ij} determines an affine connexion such that

$$g_{ij,k} = 0,$$

where the comma denotes covariant differentiation. The components of this affine connexion are given by the Christoffel formulas

$$\left\{ {i \atop jk} \right\} = g^{ai}[a,jk],$$

in which
$$[a,jk] = \frac{1}{2}\left(\frac{\partial g_{aj}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^a}\right).$$

4. **The projective connexion Γ.** There are two projective connexions which are determined in a formally obvious way by the tensor $G_{\alpha\beta}$, namely, the projective connexion Γ such that the projective derivative of $\gamma_{\alpha\beta}$ with respect to it is zero, and the projective connexion Π such that the projective derivative of $G_{\alpha\beta}$ with respect to it is zero. In this section we shall derive some of the formulas relating to Γ. We indicate projective differentiation with respect to Γ by a semicolon, so that

$$\gamma_{\alpha\beta;\gamma} \equiv \frac{\partial \gamma_{\alpha\beta}}{\partial x^\gamma} - \gamma_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma} - \gamma_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma} = 0. \tag{4.1}$$

These equations can be solved in the usual way to give

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2}\gamma^{\alpha\sigma}\left(\frac{\partial \gamma_{\sigma\beta}}{\partial x^\gamma} + \frac{\partial \gamma_{\sigma\gamma}}{\partial x^\beta} - \frac{\partial \gamma_{\beta\gamma}}{\partial x^\sigma}\right) \tag{4.2}$$

$$= \tfrac{1}{2}\gamma^{\alpha\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^\gamma} + \frac{\partial g_{\sigma\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\sigma}\right) +$$

$$+ \tfrac{1}{2}\delta^\alpha_0\left(\frac{\partial \phi_\beta}{\partial x^\gamma} + \frac{\partial \phi_\gamma}{\partial x^\beta}\right) + \gamma^{\alpha\sigma}(\phi_{\sigma\gamma}\phi_\beta + \phi_{\sigma\beta}\phi_\gamma),$$

$\phi_{\sigma\gamma}$ being defined by (3.7). It is immediately obvious on using formulas such as (3.6) that

$$\Gamma^{\alpha}_{\alpha\beta} = \{^{a}_{ab}\}\,\delta^{b}_{\beta}. \tag{4.3}$$

If we substitute $\beta = 0$ and $\gamma = 0$ in (4.2), it is clear that

$$\Gamma^{\alpha}_{00} = 0. \tag{4.4}$$

Making the same substitution in (4.1) we have

$$\phi_{\sigma}\,\Gamma^{\sigma}_{\alpha 0} = 0. \tag{4.5}$$

Substituting $\beta = 0$ and $\gamma = 0$ separately in (4.1) we have

$$\phi_{\alpha;\gamma} \equiv \frac{\partial \phi_{\alpha}}{\partial x^{\gamma}} - \phi_{\sigma}\,\Gamma^{\sigma}_{\alpha\gamma} = \gamma_{\alpha\sigma}\,\Gamma^{\sigma}_{0\gamma}$$

and

$$\gamma_{\sigma\beta}\,\Gamma^{\sigma}_{\alpha 0} = -\gamma_{\alpha\sigma}\,\Gamma^{\sigma}_{\beta 0}.$$

From these two sets of equations together it follows that

$$\phi_{\alpha;\gamma} = -\phi_{\gamma;\alpha}$$

and hence that*

$$\phi_{\alpha;\beta} = \phi_{\alpha\beta}. \tag{4.6}$$

Returning to (4.2) we find

$$\Gamma^{\lambda}_{\gamma 0} = \gamma^{\alpha\lambda}\phi_{\alpha\gamma} = \gamma^{i\lambda}\phi_{i\gamma}, \tag{4.7}$$

which yield

$$\Gamma^{i}_{j0} = \phi^{i}_{j} \quad \text{and} \quad \Gamma^{0}_{j0} = -\phi_{i}\,\phi^{i}_{j}. \tag{4.8}$$

It also follows from (4.2) that

$$\Gamma^{i}_{jk} = \{^{i}_{jk}\} + (\phi^{i}_{j}\phi_{k} + \phi^{i}_{k}\phi_{j}) \tag{4.9}$$

and

$$\Gamma^{0}_{jk} = -\phi_{i}\Gamma^{i}_{jk} + \frac{1}{2}\left(\frac{\partial \phi_{j}}{\partial x^{k}} + \frac{\partial \phi_{k}}{\partial x^{j}}\right). \tag{4.10}$$

The projective curvature tensor (cf. G.P., p. 156) for the connexion Γ is given by the formula

$$B^{\alpha}_{\beta\gamma\delta} = \frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} - \frac{\partial \Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} + \Gamma^{\xi}_{\beta\gamma}\,\Gamma^{\alpha}_{\epsilon\delta} - \Gamma^{\xi}_{\beta\delta}\,\Gamma^{\alpha}_{\epsilon\gamma}. \tag{4.11}$$

Since

$$\Gamma^{\alpha}_{\beta 0} \neq \delta^{\alpha}_{\beta},$$

its components do not vanish whenever one of the subscripts is zero, but it does satisfy

$$B^{\alpha}_{\beta 00} = 0. \tag{4.12}$$

Hence the following equation defines an affine tensor

$$B^{i}_{0j0} = \phi^{a}_{j}\,\phi^{i}_{a}. \tag{4.13}$$

On contracting (4.11) with respect to α and γ we obtain a projective tensor analogous to the Ricci tensor,

$$B_{\alpha\beta} = R_{ij}\delta^{i}_{\alpha}\delta^{j}_{\beta} - \phi^{s}_{i,s}(\delta^{i}_{\alpha}\phi_{\beta} + \delta^{i}_{\beta}\phi_{\alpha}) + 2\phi^{s}_{\beta}\phi_{s\alpha} + \phi^{s}_{t}\phi^{t}_{s}\phi_{\alpha}\phi_{\beta}, \tag{4.14}$$

* For this relation I am indebted to Dr. W. P. Allis.

where R_{ij} represents the Ricci tensor (contracted curvature tensor) formed from $\{^{i}_{jk}\}$. From this we get at once the scalar,

$$B = \gamma^{\alpha\beta} B_{\alpha\beta} = R - \phi^s_i \phi^t_s, \tag{4.15}$$

where R is the scalar curvature. The formula (4.14) also yields the affine scalar

$$B_{00} = \phi^s_i \phi^t_s, \tag{4.16}$$

the affine tensor

$$B^{ij} = R^{ij} + 2g^{st}\phi^i_s \phi^j_t, \tag{4.17}$$

and the affine vector

$$B^i_0 = -g^{ij}\phi^s_{j,s} = \frac{1}{\sqrt{g}} \frac{\partial(\phi^{is}\sqrt{g})}{\partial x^s}. \tag{4.18}$$

5. **The projective connexion Π.** The other projective connexion referred to at the beginning of section 4 is the one for which

$$G_{\alpha\beta/\gamma} \equiv \frac{\partial G_{\alpha\beta}}{\partial x^\gamma} - G_{\alpha\sigma}\Pi^\sigma_{\beta\gamma} - G_{\sigma\beta}\Pi^\sigma_{\alpha\gamma} = 0. \tag{5.1}$$

The solidus will be used to denote projective differentiation with respect to Π. For $\gamma = 0$, (5.1) gives the relation,

$$\gamma_{\alpha\sigma}\Pi^\sigma_{\beta 0} + \gamma_{\sigma\beta}\Pi^\sigma_{\alpha 0} = 2N\gamma_{\alpha\beta}. \tag{5.2}$$

On account of the relation

$$G_{\alpha\beta} = \Phi^2 \gamma_{\alpha\beta},$$

the formal work of computing the components of Π will be similar to that which arises in conformal geometry. Expanding equations (5.1) we have

$$2\Phi\Phi_{/\gamma}\gamma_{\alpha\beta} + \Phi^2\gamma_{\alpha\beta/\gamma} = 0.$$

We may write

$$\Phi_\lambda = \frac{1}{N}\frac{\partial \log \Phi}{\partial x^\lambda}, \tag{5.3}$$

the factor $1/N$ being introduced so that we shall have

$$\Phi_0 = 1,$$

and the equations defining Π reduce to

$$\gamma_{\alpha\beta/\gamma} + 2N\gamma_{\alpha\beta}\Phi_\gamma = 0. \tag{5.4}$$

Now we use the fact that the difference between any two projective connexions is a projective tensor, and so write

$$\Pi^\alpha_{\beta\gamma} = \Gamma_{\beta\gamma} + T^\alpha_{\beta\gamma}.$$

The determination of $\Pi^\alpha_{\beta\gamma}$, therefore, can be reduced to the computation of the components of $T^\alpha_{\beta\gamma}$. If we expand (5.4) we now find

$$\left(\frac{\partial\gamma_{\alpha\beta}}{\partial x^\lambda} - \gamma_{\sigma\beta}\Gamma^\sigma_{\alpha\lambda} - \gamma_{\alpha\sigma}\Gamma^\alpha_{\beta\lambda}\right) - \gamma_{\sigma\beta}T^\sigma_{\alpha\lambda} - \gamma_{\alpha\sigma}T^\sigma_{\beta\lambda} + 2N\gamma_{\alpha\beta}\Phi_\lambda = 0.$$

The terms in parenthesis add up to zero because of (4.1). Hence we have algebraic equations for $T^{\alpha}_{\beta\gamma}$ which, solved in the usual manner, give

$$T^{\lambda}_{\alpha\beta} = N(\delta^{\lambda}_{\beta}\Phi_{\alpha} + \delta^{\lambda}_{\alpha}\Phi_{\beta} - \gamma_{\alpha\beta}\Phi^{\lambda}). \tag{5.5}$$

In some of the applications it is desirable to have particular sets of components of $\Pi^{\alpha}_{\beta\gamma}$ explicitly in terms of g, ϕ, and Φ. In order to write these formulas conveniently, we define

$$\theta_{\alpha} = \phi_{\alpha} - \Phi_{\alpha}. \tag{5.6}$$

Thus $\theta_0 = 0$, and θ_i is an affine vector, and we shall raise and lower its indices by means of the g's. From

$$\Phi^{\sigma} = \gamma^{\sigma 0} + \gamma^{\sigma i}\Phi_i$$

it follows by (3.8), (3.9), and (3.10) that

$$\Phi^{i} = -\theta^{i} \text{ and } \Phi^{0} = 1 + \theta^{i}\phi_i. \tag{5.7}$$

In like manner,

$$\Phi^{\alpha}\Phi_{\alpha} = 1 + \theta^{i}\theta_i. \tag{5.8}$$

Thus we find

$$T^{0}_{00} = N(1 - \theta^{i}\phi_i), \qquad T^{i}_{00} = N\theta^{i}, \tag{5.9a}$$

$$T^{0}_{j0} = -N(\theta_j + \phi_j\,\phi_i\,\theta^{i}), \qquad T^{i}_{j0} = N(\delta^{i}_{j} + \theta^{i}\phi_j), \tag{5.9b}$$

$$T^{0}_{jk} = -N(g_{jk} + \phi_j\,\phi_k)(1 + \theta^{i}\phi_i), \tag{5.9c}$$

$$T^{i}_{jk} = N[\delta^{i}_{j}\Phi_k + \delta^{i}_{k}\Phi_j + \theta^{i}(g_{jk} + \phi_j\,\phi_k)]. \tag{5.9d}$$

There are also some formulas connecting projective derivatives with respect to Γ with covariant derivatives with respect to $\{^{i}_{jk}\}$. Thus, since $\Phi_{\alpha;\beta} = \dfrac{\partial\Phi_{\alpha}}{\partial x^{\beta}} - \Gamma^{\sigma}_{\alpha\beta}\Phi_{\sigma}$, we get

$$\Phi_{0;0} = 0. \tag{5.10}$$

Also by reference to (4.8)

$$\Phi^{\alpha}_{;\alpha} = \frac{\partial\Phi^{\alpha}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\sigma\alpha}\Phi^{\sigma} = -\frac{\partial\theta^{i}}{\partial x^{i}} - \{^{j}_{ji}\}\theta^{i}.$$

Thus, if we use commas to denote covariant derivation with respect to $\{^{i}_{jk}\}$,

$$\Phi^{\alpha}_{;\alpha} = -\theta^{i}_{,i} = -\frac{1}{\sqrt{g}}\frac{\partial(\theta^{i}\sqrt{g})}{\partial x^{i}}. \tag{5.11}$$

Again, using (4.8)

$$\Phi^{i}_{;0} = -\phi^{i}_{j}\theta^{j}. \tag{5.12}$$

And also,

$$\gamma^{\sigma j}\Phi^{i}_{;\sigma} = g^{ja}(\phi^{i}_{a} - \theta^{i}_{,a}) = -\tfrac{1}{2}g^{ia}g^{jb}(\theta_{a,b} + \theta_{b,a}). \tag{5.13}$$

We now calculate the curvature tensor formed out of the $\Pi^{\alpha}_{\beta\gamma}$ which we shall denote by $P^{\alpha}_{\beta\gamma\delta}$. Since $\Pi^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + T^{\alpha}_{\beta\gamma}$, we have

$$P^{\alpha}_{\beta\gamma\delta} = B^{\alpha}_{\beta\gamma\delta} + T^{\alpha}_{\beta\gamma;\delta} - T^{\alpha}_{\beta\delta;\gamma} + T^{\sigma}_{\beta\gamma}T^{\alpha}_{\sigma\delta} - T^{\sigma}_{\beta\delta}T^{\alpha}_{\sigma\gamma}, \tag{5.14}$$

where $B^{\alpha}_{\beta\gamma\delta}$ is, as previously defined, the curvature tensor formed from the $\Gamma^{\alpha}_{\beta\gamma}$, and $T^{\alpha}_{\beta\gamma\delta}$ denotes the covariant derivative of $T^{\alpha}_{\beta\gamma}$ with respect to the $\Gamma^{\alpha}_{\beta\gamma}$. Using (5.5) we have

$$\begin{aligned} P^{\alpha}_{\beta\gamma\delta} = B^{\alpha}_{\beta\gamma\delta} &+ N\delta^{\alpha}_{\gamma}\{\Phi_{\beta;\delta} - N\Phi_{\beta}\Phi_{\delta} + N\Phi^{\sigma}\Phi_{\sigma}\gamma_{\beta\delta}\} \\ &- N\delta^{\alpha}_{\delta}\{\Phi_{\beta;\gamma} - N\Phi_{\beta}\Phi_{\gamma} + N\Phi^{\sigma}\Phi_{\sigma}\gamma_{\beta\gamma}\} \\ &- N\gamma_{\beta\gamma}\{\Phi^{\alpha}_{;\delta} - \Phi^{\alpha}\Phi_{\delta}\} \\ &+ N\gamma_{\beta\delta}\{\Phi^{\alpha}_{;\gamma} - \Phi^{\alpha}\Phi_{\gamma}\}. \end{aligned} \tag{5.15}$$

On contracting with respect to α and γ we obtain the generalized Ricci tensor,

$$P_{\beta\delta} = B_{\beta\delta} + N(n-1)\{\Phi_{\beta;\delta} - N\Phi_{\beta}\Phi_{\delta} + N\Phi^{\sigma}\Phi_{\sigma}\gamma_{\beta\delta}\} + N\gamma_{\beta\delta}\Phi^{\sigma}_{;\sigma}, \tag{5.16}$$

and the scalar,

$$P = B - 2Nn\theta^{i}_{,i} + N^{2}n(n-1)(1+\theta^{i}\theta_{i}). \tag{5.17}$$

From (5.16) we also get the affine invariants,

$$\begin{aligned} P^{ij} &= \gamma^{i\alpha}\gamma^{j\beta}P_{\alpha\beta} \\ &= B^{ij} + N^{2}(n-1)[g^{ij}(1+\theta_{p}\theta_{p}) - \theta^{i}\theta^{j}] - \\ &\qquad - \tfrac{1}{2}N(n-1)g^{i\alpha}g^{jb}(\theta_{a,b} + \theta_{b,a}) - Ng^{ij}\theta^{p}_{,p}, \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} P^{i}_{0} &= \gamma^{i\alpha}P_{\alpha 0} \\ &= B^{i}_{0} + N(n-1)[N\theta^{i} - \phi^{i}_{j}\theta^{j}], \end{aligned} \tag{5.19}$$

and

$$P_{00} = B_{00} + N^{2}(n-1)\theta_{i}\theta^{i} - N\theta^{i}_{,i}. \tag{5.20}$$

6. **The flat case.** The flat projective geometry is the theory of a projective connexion which in a particular representation (cf. G.P., p. 159) has the components

$$\Pi^{\alpha}_{\beta\gamma} = \delta^{\alpha}_{\beta}\delta^{0}_{\gamma} + \delta^{\alpha}_{\gamma}\delta^{0}_{\beta} - \delta^{\alpha}_{0}\delta^{0}_{\beta}\delta^{0}_{\gamma}. \tag{6.1}$$

The differential equations

$$\frac{\partial X^{\alpha}}{\partial q^{\beta}} + \Pi^{\alpha}_{\sigma\beta}X^{\sigma} = 0 \tag{6.2}$$

are completely integrable and have solutions of the form

$$X^{\alpha} = p^{\alpha}_{\beta}(q)X^{\beta}_{0}, \tag{6.3}$$

in which the X_0's are arbitrary constants multiplied by $\exp(-q^0)$. The X_0's can be regarded as homogeneous co-ordinates in the tangent space at a fixed point, say $y = q_0$, of the underlying space and the X's as homogeneous co-ordinates in the tangent space at a variable point, $y = q$. The equations (6.3) then represent a projective transformation from the one tangent space to the other.

But, as is proved at the end of G.P., the X_0's and X's which satisfy (6.3) can be expressed as functions of y, each such function satisfying

$$\frac{\partial^2 X}{\partial y^\alpha \partial y^\beta} - \Pi^\sigma_{\alpha\beta} \frac{\partial X}{\partial y^\sigma} = 0. \tag{6.4}$$

In other words, if $n+1$ independent functions X_0 satisfy these equations, they define a transformation from the underlying space to the tangent space at $y = q_0$ and, moreover, the functions X related to X_0 by (6.3) also satisfy (6.4) and define a transformation from the underlying space to the tangent space at $y = q$. Hence the X_0's and the X's can both be regarded as homogeneous co-ordinates for the underlying space, and (6.3) thus defines a projective transformation of the underlying space into itself.

This may be summarized by saying that in the flat projective geometry all the tangent spaces can be regarded as coinciding with one another and with the underlying space.

If we take q_0 to be $(0\ \ 0\ \ \ldots\ 0)$ we find

$$X^0_0 = e^{y^0} \text{ and } X^i_0 = e^{y^0} y^i \tag{6.5}$$

as solutions of (6.4), and the co-ordinates y are cartesian. The equation of a quadric in cartesian co-ordinates is

$$a_{ij} y^i y^j + 2a_{0j} y^j + a_{00} = 0. \tag{6.6}$$

In the homogeneous co-ordinates X_0 this becomes

$$a_{\alpha\beta} X^\alpha_0 X^\beta_0 = 0. \tag{6.7}$$

More generally, we can replace (6.5) by

$$X^0 = e^{y^0 - q^0} \text{ and } X^i = (y^i - q^i) e^{y^0 - q^0}, \tag{6.8}$$

or

$$X^0 = X^0_0 e^{-q^0} \text{ and } X^i = (X^i_0 - q^i X^0_0) e^{-q^0}.$$

When the X's are defined in this way, which is a special case of (6.3), they satisfy (6.2) as functions of q and (6.4) as functions of y. The X's can be regarded as homogeneous co-ordinates in the tangent space associated with the point q and factor q^0—or, if we like, as homogeneous co-ordinates in the underlying space, so chosen that the point whose homogeneous co-ordinates are proportional to δ^α_0 has cartesian co-ordinates $(q^1\ \ q^2\ \ \ldots\ \ q^n)$.

The equation of the quadric (6.7) is

$$a_{ij} y^i y^j + 2a_{i0} y^i + a_{00} = a_{ij}(y^i - q^i)(y^j - q^j) + 2(a_{i0} + a_{is} q^s)(y^i - q^i) + \\ + (a_{00} + 2a_{i0} q^i + a_{ij} q^i q^j) = 0.$$

Hence in the variable homogeneous co-ordinate system defined by (6.8) the equation of the quadric is

$$G_{\alpha\beta} X^\alpha X^\beta = 0 \tag{6.9}$$

in which

$$\begin{cases} G_{ij} = e^{2q^0} a_{ij}, & G_{i0} = e^{2q^0}(a_{i0} + a_{ij} q^j), \\ G_{00} = e^{2q^0}(a_{00} + 2a_{i0} q^i + a_{ij} q^i q^j). \end{cases} \tag{6.10}$$

The X^α's as functions of q are the components of projective vectors of index -1. Hence the left-hand member of (6.9) will be invariant, if the functions $G_{\alpha\beta}$ are the components of a projective tensor of index 2. This tensor in each representation gives us the coefficients of the equation of the quadric in the projective co-ordinates associated with the point q.

It is easy to verify that the projective derivative of the projective tensor $G_{\alpha\beta}$ is zero, i.e. that

$$\frac{\partial G_{\alpha\beta}}{\partial x^\gamma} - G_{\alpha\sigma} \Pi^\sigma_{\beta\gamma} - G_{\sigma\beta} \Pi^\sigma_{\alpha\gamma} = 0. \tag{6.11}$$

We have merely to substitute (6.10) and (6.1) in these equations. Thus we see that the theory of a quadric in flat projective geometry may be regarded as the theory of a projective tensor of degree 2 and index 2 whose projective derivative with respect to a flat projective connexion is zero.*

Since the projective connexion Π of section 5 is now given in one representation by (6.1), we infer from (4.4) and (5.9a) that

$$\theta^i = 0 \text{ and } \phi_\alpha = \Phi_\alpha. \tag{6.12}$$

Hence the equation of the quadric in the homogeneous co-ordinates associated with any point $x = q$ can be written

$$g_{ij} X^i X^j + (\Phi_\alpha X^\alpha)^2 = 0,$$

which corresponds to the fact that

$$\Phi_\alpha X^\alpha = 0$$

* Obviously the equation (6.9) can be multiplied through by any factor of the form $\exp(Mq^0)$ without affecting its validity as an equation of the quadric. Therefore the quadric can be represented by a projective tensor of degree 2 and any index. But, if the index is not 2, the equation (6.11) must be replaced by

$$G_{\alpha\beta/\gamma} - G_{\alpha\beta/0} B_\gamma = 0, \tag{6.11a}$$

in which B_γ is the projective vector whose components, in the representation for which the components of Π are given by (6.1), are

$$B_\gamma = \delta^0_\gamma.$$

is the equation of the hyperplane polar to $x = q$, and

$$g_{ij}X^i X^j = 0$$

is the equation of the tangent cone through $x = q$.

Moreover, $$g_{ij}dq^i dq^j$$

is the fundamental differential form of the non-Euclidean metric with the quadric as absolute. This is to be expected, because this metric is determined in an invariant way by the quadric. It can be proved in the following manner. Consider the cartesian co-ordinates

$$z^i = \frac{X^i}{\Phi_\alpha X^\alpha}.$$

In these co-ordinates the equation of the quadric is

$$g_{ij}z^i z^j + 1 = 0.$$

Hence the quadric is a sphere of radius 1 in a Euclidean space whose hyperplane at infinity is the polar of the point $z = 0$. The fundamental tensor of the Euclidean space is g_{ij} at $z = 0$ and its components are constant in the co-ordinate system z. Thus a cross-ratio of two points $z^i = 0$ and $z^i = dx^i$ with the two points in which the line joining them meets the quadric is

$$R = \frac{1+\sqrt{g_{ij}\,dx^i dx^j}}{1-\sqrt{g_{ij}\,dx^i dx^j}}.$$

The Cayley definition of non-Euclidean distance is $\gamma \log R$ which gives

$$ds^2 = 4\gamma^2 g_{ij}dq^i dq^j$$

for the element of distance. Hence g_{ij} is the fundamental tensor of the non-Euclidean metric with the quadric as absolute; and this metric at any point is identical with a Euclidean metric whose absolute is the intersection of the polar hyperplane of the point with the quadric.

The values of the Christoffel symbols of the non-Euclidean metric come out at once from the general theory. For by (4.9) and (5.9d) we have

$$0 = \Pi^i_{jk} = \{^i_{ik}\} + (\delta^i_j \Phi_k + \delta^i_k \Phi_j)$$

and in this representation,

$$\Phi = e^{q^0}\sqrt{a_{00} + 2a_{i0}q^i + a_{ij}q^i q^j}.$$

In like manner we deduce from (4.10) and (5.9c) that

$$0 = \Pi^0_{jk} = \frac{\partial \Phi_j}{\partial x^k} - \Phi_i \left\{ {i \atop jk} \right\} - g_{jk} - \Phi_j \Phi_k$$

or

$$g_{jk} = \frac{\partial \Phi_j}{\partial x^k} + \Phi_j \Phi_k = \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial x^j \partial x^k}.$$

These remarks on the non-Euclidean geometry should be compared with those of Mr. Whitehead in P.F.

7. **Infinitesimal projective displacements.** Any projective connexion can be interpreted as determining an infinitesimal displacement of the tangent spaces of the sort studied by Cartan, Schouten, and others. We have explained this for the flat case in the section above as an interpretation of the equations

$$X^\alpha_{/\beta} = 0, \tag{7.1}$$

in which X^α is of index -1. The same equations can be used for the geometry-of-paths case (cf. G.P., p. 150 et seq.), that is, for any connexion Λ such that

$$\Lambda^\alpha_{\beta 0} = \delta^\alpha_\beta. \tag{7.2}$$

But in general the projective connexions Γ and Π do not satisfy (7.1) when $\beta = 0$. We must therefore use another set of equations, if we are to interpret these connexions in terms of projective displacements. We turn to the equations

$$X_{/} = X^\alpha_{/0} B_\beta, \tag{7.3}$$

in which B_β is a projective vector such that

$$B_0 = 1.$$

The latter condition insures that the equations shall be satisfied when $\beta = 0$. The equations (7.3) expand into

$$\frac{\partial X^\alpha}{\partial x^\beta} + \Pi^\alpha_{\sigma\beta} X^\sigma + B_\beta X^\alpha - \Pi^\alpha_{\sigma 0} X^\sigma B_\beta = 0$$

which are of the form

$$\frac{\partial X^\alpha}{\partial x^\beta} + \Lambda^\alpha_{\sigma\beta} X^\sigma = 0, \tag{7.4}$$

where

$$\Lambda^\alpha_{\sigma\beta} = \Pi^\alpha_{\sigma\beta} - \Pi^\alpha_{\sigma 0} B_\beta + \delta^\alpha_\sigma B_\beta. \tag{7.5}$$

The quantities $\Lambda^\alpha_{\sigma\beta}$ are the components of an unsymmetric projective connexion for which (7.2) is satisfied.

In general, the equations (7.4) are not integrable. But, along an arbitrary curve $x^i = x^i(t)$, we have

$$\frac{dX^\alpha}{dt} + \Lambda^\alpha_{\sigma j} \frac{dx^j}{dt} X^\sigma = 0$$

which, being linear, integrates into

$$X^\alpha = P^\alpha_\beta(t) X^\beta_0, \tag{7.6}$$

where X^β_0 are arbitrary constants. We may interpret X^β_0 as homogeneous co-ordinates in the tangent space at the point $t=0$ and X^α as homogeneous co-ordinates in the tangent space at any other point of the curve. The equations (7.6) then represent a projective transformation of the first tangent space into the second. In this sense we may say that

$$dX^\alpha + \Lambda^\alpha_{\sigma j} X^\sigma dx^j = 0 \tag{7.7}$$

represents an infinitesimal projective transformation of the tangent space at any point x into that at any point $x+dx$.

The equations (7.7) represent a single infinitesimal transformation —not, as one might expect, an infinite family of them. For, if we make a change of factor, the components $\Lambda^\alpha_{\beta\gamma}$ are changed, but so also are the components X^α which represent a point of the tangent space, and the expressions on the left-hand side of (7.7) are transformed as the components of a projective vector.

Let us write (7.7), using V for the non-homogeneous co-ordinates of any point in the tangent space and dx for the co-ordinates of the displacement. So

$$V^i = AX^i = \frac{X^i}{A_\alpha X^\alpha},$$

and (7.7) becomes

$$dV^i + (\Lambda^i_{kj} - \Lambda^i_{0j} A_k - A_\alpha \Lambda^\alpha_{0j} \delta^i_k) V^k dx^j + \tag{7.8}$$
$$+ V^i \left(\frac{\partial A_k}{\partial x^j} - A_\alpha \Lambda^\alpha_{kj} + A_k A_\alpha \Lambda^\alpha_{0j} \right) V^k dx^j + \Lambda^i_{0j}\, dx^j = 0.$$

It can be verified directly, if desired, that this expression is unaltered by a change of factor.

It is always possible to choose the factor so that $A_0 = 1$ and $A_t = 0$. In this representation (7.8) becomes

$$dV^i + \Lambda^i_{kj} V^k dx^j - \Lambda^0_{0j} V^i dx^j - \Lambda^0_{kj} V^i V^k dx^j + \Lambda^i_{0j}\, dx^j = 0. \tag{7.9}$$

We now ask what happens to the quadrics

$$G_{\alpha\beta} X^\alpha X^\beta = 0, \tag{7.10}$$

when the tangent spaces undergo the projective displacements determined by (7.7). If we differentiate the left-hand member of (7.10) and substitute (7.4), we obtain

$$\left(\frac{\partial G_{\alpha\beta}}{\partial x^\gamma} - G_{\sigma\beta} \Lambda^\sigma_{\alpha\gamma} - G_{\alpha\sigma} \Lambda^\sigma_{\beta\gamma} \right) X^\alpha X^\beta.$$

Any one of the quadrics will be transformed into another one, if these quantities vanish, when and only when (7.10) is satisfied. This will happen if

$$\frac{\partial G_{\alpha\beta}}{\partial x^\gamma} - G_{\sigma\beta}\Lambda^\sigma_{\alpha\gamma} - G_{\alpha\sigma}\Lambda^\sigma_{\beta\gamma} = G_{\alpha\beta}C_\gamma \tag{7.11}$$

for some projective vector C_γ. On substituting (7.4) in these equations and using (5.1) and (5.2), we obtain

$$2(N-1)B_\gamma = C_\gamma,$$

which determines C_γ so that (7.11) is satisfied. Hence any tangent quadric is transformed into another by the infinitesimal projective transformations determined by Π according to the formulas (7.3).

The discussion in this section up to (7.11) can be carried through unaltered, if the projective differentiation in (7.3) is taken to be with respect to Γ instead of Π. Let us, then, substitute the equation like (7.5), with Γ replacing Π, in (7.11) and obtain

$$-2B_\gamma = C_\gamma.$$

Hence the tangent quadrics are transformed into themselves by the infinitesimal projective transformation determined by Γ according to the equations

$$X^\alpha_{i\gamma} = X^\alpha_{i0}\,B_\gamma.$$